Some Elements on Lévy Processes

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8th November 2010
Outline

1 Introduction
   Basic Definitions
   Poisson Process etc.
2 Basic Aspects on Lévy Processes
   Famous Processes
   Main Properties
   Examples
3 Structure of Lévy Processes
   Jump Process
   Decomposition of a Lévy Process
4 Some Sample Path Properties
   Recurrence and transience
5 Stock Model with Jumps
   Jump Diffusion
Some Elements on Lévy Processes

Outline

Introduction
- Basic Definitions
- Poisson Process etc.

Basic Aspects on Lévy Processes
- Famous Processes
- Main Properties
- Examples

Structure of Lévy Processes
- Jump Process
- Decomposition of a Lévy Process

Some Sample Path Properties
- Recurrence and transience

Stock Model with Jumps
- Jump Diffusion

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Introduction
Lévy processes

= processes in continuous time with independent and stationary increments

- Important class of Markov processes.
- Natural examples of semimartingales for which stochastic calculus applies.
- Appeared in physics: problems in turbulence, laser cooling.
- Important role in mathematical finance (heavy tails).

Paul Pierre Lévy
(1886-1971)
Filtered Probability Space

\((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\)

Filtration \((\mathcal{F}_t)_{t \geq 0}\) fulfills the standard conditions:

- \(\mathcal{F}_s \subseteq \mathcal{F}_t\) for \(s \leq t\) (as times moves forward, we obtain more and more information).
- Filtration is right-continuous, i.e. \(\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}\).

\(X_t\) is an \(\mathcal{F}_t\)-adapted stochastic process, if \(\sigma(X_t) \subseteq \mathcal{F}_t, \forall t \geq 0\) (\(X_t\) is \(\mathcal{F}_t\)-measurable for each \(t\)).

Filtration models the flow of public information. Price processes are adapted to this filtration (i.e. the filtration contains the observed history of market variables).
**Stopping Time (Markův čas)**

**Definition 1**
Suppose a \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\). A stopping time \(\tau\) is a random variable taking values in \([0, \infty]\) and satisfying

\[
[\tau \leq t] \in \mathcal{F}_t, \quad \forall t \geq 0.
\]

**Properties:** \([\tau = t] \in \mathcal{F}_t\), i.e. the decision to stop at time \(t\) is based on information available at time \(t\).

**Definition 2**
We have an adapted process \(X_t\) and a stopping time \(\tau\). The stopped process is defined as

\[
X_{t \wedge \tau} = \begin{cases} 
X_t & t \leq \tau \\
X_\tau & t > \tau 
\end{cases}
\]
### Stopping Time

**Examples:**

Hitting time of a one-sided boundary by a Brownian motion

\[ \tau_a := \inf\{ t \geq 0 : W_t = a \} \]
Poisson Process: Construction

\[ \tau_1, \tau_2, \ldots \sim \text{Exp}(\lambda) \text{ exponentially i.i.d. random variables,} \]
i.e. \( f(t) = \lambda e^{-\lambda t}, \quad t \geq 0, \quad E\tau_i = 1/\lambda. \)

Let \( S_n \) be the time of the \( n \)-th jump

\[ S_n = \sum_{k=1}^{n} \tau_k. \]

Poisson process with intensity \( \lambda \) is the number of jumps at or before time \( t \)

\[ N_t = \sum_{i=1}^{\infty} I[S_i \leq t] \]

- \( N_t \) is right-continuous
- \( N_t \) is not predictable w.r.t. \( \mathcal{F}_t \), i.e. not \( \mathcal{F}_{t^-} \) measurable
Poisson Process: Basic Properties

Poisson process jumps are of size 1.

**Lemma 3**

*The Poisson process* $N_t$ *with intensity* $\lambda > 0$ *has the Poisson distribution* $\text{Po}(\lambda t)$

$$
P[N_t = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, \ldots
$$

Memorylessnes of the exponential distribution

$\Rightarrow$ Poisson process is memoryless

For $t, s \geq 0$

$$
\mathcal{L}(N_{t+s} - N_s) = \mathcal{L}(N_t) = \text{Po}(\lambda t)
$$
Poisson Process: Basic Properties

Mean and variance of the Poisson process:

\[ \mathbb{E} N_t = \lambda t \]
\[ \text{Var}(N_t) = \lambda t \]

Theorem 4

The compensated Poisson process defined as

\[ M_t = N_t - \lambda t \]

is a martingale.
Compound Poisson Process

... to allow the jump sizes to be random

\[ \xi_1, \xi_2, \ldots \text{ i.i.d. with } \beta = \mathbb{E}\xi_i \text{ independent of Poisson process } N_t \]

**Compound Poisson process**

\[ Y_t := \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0. \]

The compound Poisson process is memoryless, increments are independent and

\[ \mathcal{L}(Y_{t+s} - Y_s) = \mathcal{L}(Y_t). \]
Compound Poisson Process

Mean of the compound Poisson process:

\[
EY_t = \mathbb{E} \left[ \sum_{i=1}^{N_t} \xi_i \right] = \sum_{k=0}^{\infty} \mathbb{E} \left[ \sum_{i=1}^{k} \xi_i \mid N_t = k \right] P[N_t = k] = \\
= \mathbb{E}\xi_1 \sum_{k=0}^{\infty} k P[N_t = k] = \mathbb{E}\xi_1 EN_t = \beta\lambda t
\]

**Theorem 5**

The compensated compound Poisson process defined as

\[ M_t = Y_t - \beta\lambda t \]

is a martingale.
Basic Aspects on Lévy Processes
Wiener Process

Definition 6
An $\mathcal{F}_t$-adapted stochastic process $X = (X_t, t \geq 0)$ with values in $\mathbb{R}$ is said to be a \textit{Wiener process}, if $\forall s, t \geq 0$

1. $W_0 = 0$ almost surely.

2. \textbf{Independent increments:} $W_{t+s} - W_t$ is independent of $\mathcal{F}_t$.

3. \textbf{Normal increments:} $W_{t+s} - W_t \sim \mathcal{N}(0, s)$.

4. Sample paths are \textit{continuous}. 
Poisson Process

Definition 7
An $\mathcal{F}_t$-adapted stochastic counting process $(N_t, t \geq 0)$ with values in $\mathbb{N}$ is said to be a Poisson process, if $\forall s, t \geq 0$

1. $N_0 = 0$ almost surely.
2. Independent increments: $N_{t+s} - N_t$ is independent of $\mathcal{F}_t$.
3. Stationary increments: $N_{t+s} - N_t$ has the same distribution as $N_s$.
4. No counted occurrences are simultaneous.
Basic Aspects on Lévy Processes

Lévy Process

Definition 8
An $\mathcal{F}_t$-adapted stochastic process $X = (X_t, t \geq 0)$ with values in $\mathbb{R}^d$ is said to be a Lévy process, if $\forall s, t \geq 0$

1. $X_0 = 0$ almost surely.
2. Independent increments: $X_{t+s} - X_t$ is independent of $\mathcal{F}_t$.
3. Stationary increments: $X_{t+s} - X_t$ has the same distribution as $X_s$.
4. Sample paths are right-continuous and possess left limits.
Càdlàg Process

= everywhere right continuous and has left limits everywhere

càdlàg: ”continu à droite, limite à gauche”

RCLL: ”right continuous with left limits”

corlol: ”continuous on (the) right, limit on (the) left”

Skorokhod space = the collection of càdlàg functions on a given domain.

- Lévy process is càdlàg.
- Continuous process is càdlàg.
Basic Aspects on Lévy Processes

Markov Property

From the properties of Lévy process we get immediately

- \(X_{t+s} \mid X_t = x\) is independent of \(\mathcal{F}_t\), \(s, t \geq 0\).
- \(\mathcal{L}(X_{t+s} \mid X_t = x) = \mathcal{L}(x + X_s), s, t \geq 0\).

**Theorem 9**

(Markov Property) Let \(\tau\) be an \((\mathcal{F}_t)\)-stopping time, \(\tau < \infty\) a.s.

- \(X_{\tau+t} \mid X_{\tau} = x\) is independent of \(\mathcal{F}_\tau\), \(t \geq 0\).
- \(\mathcal{L}(X_{\tau+t} \mid X_{\tau} = t) = \mathcal{L}(x + X_t), t \geq 0\).

Often applied to investigate distributions related to *first passage time* (čas prvního průchodu)

\(\tau_B := \inf\{t \geq 0 : X_t \in B\}\), \(B\) is a Borel set.

\(\Rightarrow \quad \tau_B\) is a stopping time.
Basic Aspects on Lévy Processes

Infinitely Divisibility

Elementary decomposition of a Process, $n \in \mathbb{N}$:

$$X_1 = X_{\frac{1}{n}} + \left( X_{\frac{2}{n}} - X_{\frac{1}{n}} \right) + \cdots + \left( X_{\frac{n}{n}} - X_{\frac{(n-1)}{n}} \right)$$

⇒ Distributions of a LP are infinitely divisible (can be expressed as the sum of $n$ i.i.d. variables, $n \in \mathbb{N}$).

Characteristic function of an infinitely divisible variable $X_1$ can be expressed in the form

$$\mathbb{E} \left( e^{i\langle \lambda, X_1 \rangle} \right) = e^{-\Psi(\lambda)}, \quad \lambda \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product and $\Psi : \mathbb{R}^d \to \mathbb{C}$ is a continuous function with $\Psi(0) = 0$ known as the characteristic exponent of $X$. 

Law of the Whole Process

Making use of the independence, stationarity of increments and right-continuity of the sample paths we get

\[ E \left( e^{i \langle \lambda, X_t \rangle} \right) = e^{-t \Psi(\lambda)}, \quad \lambda \in \mathbb{R}^d, \ t \geq 0. \]

\[ \Rightarrow \text{the law of the Lévy process is completely determined by } \Psi. \]
Lévy-Khintchine formula

**Theorem 10**

A function $\Psi : \mathbb{R}^d \rightarrow \mathbb{C}$ is the characteristic exponent of an infinitely divisible distribution if and only if it can be expressed in the form

$$\Psi(\lambda) = -i\langle a, \lambda \rangle + \frac{1}{2} Q(\lambda) +$$

$$+ \int_{\mathbb{R}^d} \left(1 - e^{i\langle \lambda, x \rangle} + i\langle \lambda, x \rangle I_{[|x|<1]} \right) \Pi(dx),$$

where $a \in \mathbb{R}^d$, $Q$ is a positive semi-definite quadratic form on $\mathbb{R}^d$, and $\Pi$ a measure on $\mathbb{R}^d \setminus \{0\}$ with $\int (1 \wedge |x|^2) \Pi(dx) < \infty$ called Lévy measure.

Moreover, $a$, $Q$ and $\Pi$ are then uniquely determined by $\Psi$. 
Lévy-Khintchine formula

Note: This formula gives the generic form of characteristic exponents.
⇒ Key to understanding the probabilistic structure of Lévy processes.

1D-version:
\[
E \left[ e^{i\lambda X_1} \right] = e^{-\psi(\lambda)} = \\
= \exp \left( ia\lambda - \frac{1}{2} \sigma^2 \lambda^2 - \int_{\mathbb{R}} \left( 1 - e^{i\lambda x} + i\lambda x |x| < 1 \right) \Pi(dx) \right)
\]

\[\psi(\lambda) = -ia\lambda + \frac{1}{2} \sigma^2 \lambda^2 + \int_{\mathbb{R}} \left( 1 - e^{i\lambda x} + i\lambda x |x| < 1 \right) \Pi(dx)\]
**Poisson Distribution**

\( X \sim \text{Po}(c), \ c > 0: \)

\[
P(X = n) = \frac{c^n}{n!} e^{-c}
\]

**Characteristic function:**

\[
E \left[ e^{i\lambda X} \right] = \sum_{n=0}^{\infty} e^{i\lambda n} \frac{c^n}{n!} e^{-c} = \exp \left( -c(1 - e^{i\lambda}) \right)
\]

**Lévy measure:** \( \Pi(dx) = c\delta_1(dx) \) (\( \delta_1 \) is the Dirac measure at 1)

**Associated Lévy process:** *Poisson process with intensity c*
Normal Distribution

\( X \sim N(0, 1) : \)

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \]

**Characteristic function:**

\[
\mathbb{E} \left[ e^{i\lambda X} \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} e^{-\frac{x^2}{2}} = \exp \left( -\frac{\lambda^2}{2} \right)
\]

**Lévy measure:** \( \Pi(dx) = 0dx \)

**Associated Lévy process:** *Standard Brownian motion*
Cauchy Distribution

\( X \sim \text{Cauchy}: \)

\[ f(x) = \frac{1}{\pi(1 + x^2)} \]

**Characteristic function:**

\[
E \left[ e^{i\lambda X} \right] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1 + x^2} dx = \exp(-|\lambda|) = \\
= \exp \left( -\frac{1}{\pi} \int_{-\infty}^{\infty} (1 - e^{i\lambda x}) x^{-2} dx \right)
\]

**Lévy measure:** \( \Pi(dx) = \pi^{-1} x^{-2} dx \)

**Associated Lévy process:** *Standard Cauchy process*
**Gamma Distribution**

\[ X \sim \Gamma(c, 1), \quad c > 0: \]

\[
f(x) = \frac{x^{c-1}e^{-x}}{\Gamma(c)}
\]

**Characteristic function:**

\[
E \left[ e^{i\lambda X} \right] = \frac{1}{\Gamma(c)} \int_{-\infty}^{\infty} e^{i\lambda x} x^{c-1} e^{-x} \, dx = (1 - i\lambda)^{-c} = \\
= \exp \left( -c \int_{0}^{\infty} \left( 1 - e^{i\lambda x} \right) x^{-1} e^{-x} \, dx \right)
\]

**Lévy measure:** \( \Pi(dx) = c 1_{[x>0]} x^{-1} e^{-x} \, dx \)

**Associated Lévy process:** *Gamma process with shape parameter c*
Stable Distributions

\[ X \sim \text{SD}(\alpha, \beta, \gamma), \quad \alpha \in (0, 1) \cup (1, 2), \beta \in [-1, 1], \gamma > 0: \]
\( \alpha = 1 \) transform of Cauchy distribution, \( \alpha = 2 \) normal distribution

Characteristic function:
\[
E \left[ e^{i\lambda X} \right] = \exp \left( -\gamma |\lambda|^{\alpha} (1 - i\beta \text{sgn}(\lambda) \tan(\pi \alpha/2)) \right)
\]

Lévy measure: \( \Pi(dx) = \begin{cases} 
  c^+ |x|^{-\alpha-1} dx & x > 0 \\
  c^- |x|^{-\alpha-1} dx & x < 0 
\end{cases} \),

where \( c^+ \) and \( c^- \) are two nonnegative real numbers such that
\( \beta = (c^+ - c^-)/(c^+ + c^-) \)

Associated Lévy process: *Stable Lévy process with index \( \alpha \) and skewness \( \beta \)*
Structure of Lévy Processes
Jumps in Lévy Process

Left-limit of $X$ at time $t$:

$$X_{t-} = \lim_{s \to t-} X_s$$

(possible) jump:

$$\Delta X_t = X_t - X_{t-}$$

For any Borel set $\{0\} \neq B \subseteq \mathbb{R}^d$, write

$$N^B_t = \text{Card}\{s \in (0, t] : \Delta X_s \in B\}$$

for the number of jumps accomplished by $X$ before time $t$ that take values in $B$. 
Structure of Lévy Processes

Jumps $\rightarrow$ Poisson Process

Independence and stationarity of the increments of $X$

$\downarrow$

- $N_t^B$ has independent and stationary increments
- sample paths of $N_t^B$ are right-continuous and they increase by jumps of size 1

$\downarrow$

$N_t^B$ is a Poisson process with intensity $\Lambda(B)$
Lévy measure

$B_1, \ldots, B_n, \ldots$ is a countable partition of $B$ (disjoint sets, union is $B$)

⇓

$N_t^{B_1}, \ldots, N_t^{B_n}, \ldots$ are independent Poisson processes with intensities $\Lambda(B_i), i = 1, \ldots, \infty$

$$N_t^B = N_t^{B_1} + \cdots + N_t^{B_n} + \ldots$$

is a Poisson process with intensity

$$\Lambda(B) = \Lambda(B_1) + \cdots + \Lambda(B_n) + \ldots$$

⇓

$\Lambda$ is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ that gives a finite mass to the complement of any neighbourhood of the origin.
Structure of the Jumps

Theorem 11

The jump process $\Delta X = (\Delta X_t, t \geq 0)$ of a Lévy process $X$ is a Poisson point process valued in $\mathbb{R}^d$, whose characteristic measure is the Lévy measure $\Pi$.

This means that for every Borel set $B$ at a positive distance from the origin, the counting process $N^B$ is a Poisson process with intensity $\Pi(B)$, and to disjoint Borel sets correspond independent Poisson processes.
Ex.: Compound Poisson Process

Let $\Lambda$ be a finite measure on $\mathbb{R}^d$ that gives no mass to the origin.
Let $(\Delta_t, t \geq 0)$ be a Poisson point process with the characteristic finite measure $\Lambda$.

**Compound Poisson process**

$$Y_t = \sum_{0 \leq s \leq t} \Delta_s$$

is a right-continuous step process and by construction its jump process is $\Delta Y_t = \Delta_t$.

$Y_t$ is a Lévy process.
Ex.: Compound Poisson Process

We compute the characteristic function

\[
E \left( e^{i \langle \lambda, Y_1 \rangle} \right) = E \left( \exp \left\{ i \sum_{0 \leq s \leq 1} \langle \lambda, \Delta_s \rangle \right\} \right) = \\
\exp \left\{ - \int_{\mathbb{R}^d} \left( 1 - e^{i \langle \lambda, x \rangle} \right) \Lambda(dx) \right\}
\]

It is a special case of the Lévy-Khintchine formula.

Characteristic measure \( \Lambda \) of the jump process is the Lévy measure.
Structure of Lévy Processes

Decomposition of a Lévy Process

Probabilistic meaning of the LK-Formula

Decomposition of the characteristic exponent $\Psi$ of the Lévy-Khintchine formula

$$\Psi = \Psi^{(0)} + \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)},$$

where

$$\Psi^{(0)}(\lambda) = -i \langle a, \lambda \rangle$$
$$\Psi^{(1)}(\lambda) = \frac{1}{2} Q(\lambda)$$
$$\Psi^{(2)}(\lambda) = \int_{\mathbb{R}^d} \left( 1 - e^{i\langle \lambda, x \rangle} \right) I_{|x| \geq 1} \Pi(dx)$$
$$\Psi^{(3)}(\lambda) = \int_{\mathbb{R}^d} \left( 1 - e^{i\langle \lambda, x \rangle} + i \langle \lambda, x \rangle \right) I_{|x| < 1} \Pi(dx)$$

Each $\Psi^{(i)}$ is a characteristic exponent of some Lévy process.
Structure of Lévy Processes

Decomposition of a Lévy Process

Continuous Part: $\psi^{(0)}$ and $\psi^{(1)}$

**Constant drift**

is a deterministic linear process with characteristic exponent

$$\psi^{(0)}(\lambda) = -i \langle a, \lambda \rangle$$

**Brownian component**

is a linear transform of a $d$-dimensional Brownian motion with characteristic exponent

$$\psi^{(1)}(\lambda) = \frac{1}{2} Q(\lambda)$$
Large Jumps: $\psi^{(2)}$

$$\psi^{(2)}(\lambda) = \int_{\mathbb{R}^d} \left(1 - e^{i\langle \lambda, x \rangle} \right) I_{|x| \geq 1} \Pi(dx)$$

is a characteristic exponent of a compound Poisson process with Lévy measure $I_{|x| \geq 1} \Pi(dx)$.
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Outline
Introduction
Basic Definitions
Poisson Process etc.
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Famous Processes
Main Properties
Examples
Structure of Lévy Processes
Jump Process
Decomposition of a Lévy Process

Small Jumps: $\psi^{(3)}$

$$
\psi^{(3)}(\lambda) = \int_{\mathbb{R}^d} \left( 1 - e^{i\langle \lambda, x \rangle} + i \langle \lambda, x \rangle \right) I_{|x|<1} \Pi(dx)
$$

In the case when

$$
\int_{\mathbb{R}^d} |x| I_{|x|<1} \Pi(dx) < \infty,
$$

we can re-write

$$
\psi^{(3)}(\lambda) = i \langle \lambda, a' \rangle + \int_{\mathbb{R}^d} \left( 1 - e^{i\langle \lambda, x \rangle} \right) I_{|x|<1} \Pi(dx)
$$

with $a' = \int_{\mathbb{R}^d} x I_{|x|<1} \Pi(dx)$. 

Structure of Lévy Processes —— Decomposition of a Lévy Process
Structure of Lévy Processes

Small Jumps: $\psi^{(3)}$

We can consider a Poisson point process $\Delta^{(3)}$ with characteristic measure $I_{|x|<1}\Pi(dx)$.

The hypothesis $\int_{\mathbb{R}^d} |x| I_{|x|<1} \Pi(dx) < \infty$ ensures that the series $\sum_{0 \leq s \leq t} |\Delta_s^{(3)}|$ converges a.s. for every $t \geq 0$, and this enables us to set

$$Y_t^{(3)} = -a't + \sum_{0 \leq s \leq t} \Delta_s^{(3)}.$$

$Y_t^{(3)}$ is a Lévy process with characteristic exponent $\psi^{(3)}$. 
Lévy-Itô Decomposition

General Lévy Process $X$ can be decomposed as the sum of four independent Lévy processes:

$$X = Y^{(0)} + Y^{(1)} + Y^{(2)} + Y^{(3)},$$

where

- $Y^{(0)}$ is a constant drift.
- $Y^{(1)}$ is linear transform of a Brownian motion.
- $Y^{(2)}$ is a compound Poisson process with jumps of size greater than or equal to 1.
- $Y^{(3)}$ is a pure jump process with jumps of size less than 1, that is obtained as the limit of compensated compound Poisson processes.
Jump Diffusion Process

Structure of Lévy Processes

Decomposition of a Lévy Process

Jump Process

Decomposition of a Lévy Process

Jump Diffusion Process
Some Sample Path Properties
Some Sample Path Properties

Recurrence and transience

Definitions

$X$ is a Lévy process with values in $\mathbb{R}^d$

$X$ is \textit{recurrent} if

$$\liminf_{t \to \infty} |X_t| = 0 \text{ a.s.}$$

$X$ is \textit{transient} if

$$\liminf_{t \to \infty} |X_t| = \infty \text{ a.s.}$$

The \textit{potential of a Borel set} $B \subseteq \mathbb{R}^d$ is the expected time spent by the Lévy process in $B$,

$$U(B) := \int_0^\infty P(X_t \in B)\,dt = \mathbb{E} \left( \int_0^\infty 1_{\{X_t \in B\}}\,dt \right)$$
Some Sample Path Properties

Recurrence and transience

Analytic Characterization

**Theorem 12**
For $\epsilon > 0$, let $B_\epsilon$ stand for the open ball in $\mathbb{R}^d$ centered at the origin with radius $\epsilon$.

If $U(B_\epsilon) < \infty$ for some $\epsilon > 0$, then the Lévy process is transient. Otherwise the Lévy process is recurrent.

**Theorem 13**
(Chung and Fuchs test) Let $X$ be a real-valued Lévy process with finite mean $\mathbb{E}X_1 = \mu \in \mathbb{R}$.

Then $X$ is transient if $\mu \neq 0$ and recurrent if $\mu = 0$. 
Stock Model with Jumps
Simple Stock Price with Jumps

We assume the stock price follows the SDE

\[ dS = \mu Sdt + \sigma SdW + (J - 1)Sdq \]

\( W \) is the Brownian motion
\( q \) is the Poisson process independent of \( W \)

\[ dq = \begin{cases} 
0 & \text{with probability } 1 - \lambda dt \\
1 & \text{with probability } \lambda dt 
\end{cases} \]

When \( dq = 1 \), the process jumps from \( S \) to \( JS \).

The jump size \( J \) is random variable independent of the Brownian motion \( W \) and the Poisson process \( q \).
Jump Diffusion Models

+  
  - capture a real phenomenon that is missing from the Black-Scholes model.

-  
  - difficulty in parameter estimation
  - it is hard to find a numerical solution
  - impossibility of perfect risk-free hedging, only hedging "on average"
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Outline
- Literature
- Introduction
  - Basic Definitions
  - Poisson Process etc.
- Basic Aspects on Lévy Processes
  - Famous Processes
  - Main Properties
  - Examples
- Structure of Lévy Processes
  - Jump Process
  - Decomposition of a Lévy Process
- Some Sample Path Properties
  - Recurrence and transience
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  - Jump Diffusion

The End