Chapter II: Markov Jump Processes

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Notation and Markov property

\[(\Omega, \mathcal{F}, \mathbb{P})\] \ldots probability space
\[E\] \ldots finite or countable state space
\[S_0, S_1, S_2, \ldots\] \ldots times of jumps, \(S_0 := 0\)
\[T_n = S_{n+1} - S_n\] \ldots holding times, \(n \in \mathbb{N}\)
\[Y_0, Y_1, Y_2 \ldots\] \ldots visited states from state space \(E\)

**Definition (Markov Chain with Continuous Time)**

The system of random variables \(\{X_t, t \geq 0\}\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) is called Markov chain with continuous time and countable (finite) state space (Markov jump process) \(E\) if

\[
\mathbb{P}(X_t = j | X_s = i, \ldots, X_{t_1} = i_1) = \mathbb{P}(X_t = j | X_s = i)
\]

for all \(j, i, i_1, \ldots, i_n \in E\) and \(0 < t_1 < \cdots < t_n < s < t\) where

\[
\mathbb{P}(X_s = i, X_{t_n} = i_n, \ldots, X_{t_1} = i_1) > 0.
\]
Basic Characteristics

Assuming that Markov jump process is time-homogenous, i.e.

$$\mathbb{P}(X_{s+t} = j | X_s = i) = \mathbb{P}(X_t = j | X_0 = i) = \mathbb{P}_i(X_t = j) = p_{ij}^t,$$

let us denote the transition semigroup by the family

$$\{p_{ij}^t, t \geq 0, \sum_{j \in E} p_{ij}^t = 1\} = \{P^t, t \geq 0\}.$$

**Chapman-Kolmogorov equations** are analogic to discrete time Markov chains

$$p_{ij}^{t+s} = \sum_{k \in E} p_{ik}^s p_{kj}^t$$

and in the matrix form $P^{t+s} = P^t P^s$ which can be also interpreted as usual matrix multiplication.
Example of Application in Insurance

- Dead
- Ill
- Healthy

States: $S_0, S_1, S_2, S_3, S_4$

Time Points: $T_1, T_2, T_3, T_4$

Transitions between states.
Absorption State

Two possible phenomena of the Markov jump process may occur. First, the process may be absorbed at some state, for eg. state $i$. It means that there exists a last finite jump time $S_n$. Then $T_j = \infty, j = n, n+1, \ldots$ and $Y_k = i, k = n, n+1, \ldots$
Explosion Time

Second, the jumps of the process may accumulate, i.e. the process explodes. Jump times of the process are defined as

\[ S_1 = \inf\{t > 0, X_t \neq X_0\} \]
\[ S_2 = \inf\{t > S_1, X_t \neq X_{S_1}\} \]

\[ \vdots \]

\[ S_n = \inf\{t > S_{n-1}, X_t \neq X_{S_{n-1}}\} \]

and jump times can be rewritten by holding times as

\[ S_n = \sum_{k=1}^{n} T_k, \quad \xi = \sup S_n = \sum_{k=1}^{\infty} T_k, \]

Random variable \( \xi \) is called explosion time.
Example - Explosion Process

\begin{itemize}
  \item $Y_2$
  \item $Y_1$
  \item $S_0$
  \item $S_1$
  \item $S_2, S_3$
  \item $S_4$
  \item $\xi$
\end{itemize}
Intuition Matrix

**Theorem**

For every state $i \in E$ exists a limit

$$\lim_{h \to 0^+} \frac{1 - p_{ii}^h}{h} := \Lambda_i \leq \infty$$

and for every $i, j \in E$, $i \neq j$ exists a limit

$$\lim_{h \to 0^+} \frac{p_{ij}^h}{h} := \Lambda_{ij} < \infty$$

and for every $i \in E$ is

$$\sum_{i \neq j} \Lambda_{ij} \leq \Lambda_i.$$ 

Nonnegative numbers $\Lambda_{ij}$ are called transition **intensities** from the state $i$ to the state $j$, $\Lambda_{ii} = -\Lambda_i$, $\Lambda_i$ is called total intensity. The matrix $\Lambda = \{\Lambda_{ij}, i, j \in E\}$ is called **intensity matrix**.
Intensity Matrix

When state space $E$ is finite then

$$\Lambda_i = \sum_{i\neq j} \Lambda_{ij} \Rightarrow \sum_{j\in E} \Lambda_{ij} = 0.$$  

If state space $E$ is infinite then

$$\Lambda_i \geq \sum_{i\neq j} \Lambda_{ij}.$$  

**Theorem**

If $\Lambda_i = 0$ then $p_{ii}^t = 1$. If $0 < \Lambda_i < \infty$ then the holding time in state $i$ has exponential distribution with expected value equal $1/\Lambda_i$.  

The following result is of gives a necessary and sufficient condition, known as **Reuter’s condition**, for a Markov jump process to be explosive (nontrivial application are birth-death processes).

**Theorem**

A Markov jump process is nonexplosive if and only if the only nonnegative bounded solution \( k = (k_i)_{i \in E} \) to the set of equations

\[
\Lambda k = k
\]

is \( k = 0 \).
Kolmogorov Differential Equations

Let \( \Lambda \) be an intensity matrix on \( E \), \( \Lambda_i < \infty \) and \( \{X_t, t \geq 0\} \) is the Markov jump process defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \), then \( E \times E \)-matrices \( P^t \) satisfy the **backward equation**, i.e.

\[
(p^t_{ij})' = \sum_{k \in E} \Lambda_{ik} p^t_{kj} = -\Lambda_i p^t_{ij} + \sum_{k \neq i} \Lambda_{ik} p^t_{kj}
\]

\[
(P^t)' = \Lambda P^t,
\]

and **forward equation**, i.e.

\[
(p^t_{ij})' = \sum_{k \in E} p^t_{ik} \Lambda_{kj} = -p^t_{ij} \Lambda_j + \sum_{k \neq i} p^t_{ik} \Lambda_{kj}
\]

\[
(P^t)' = P^t \Lambda,
\]

assuming that \( \frac{p^h_{ij}}{h} \to \Lambda_{ij} \) converges uniformly in \( i \).
Kolmogorov Differential Equations

Theorem

If $E$ is finite and $\Lambda = \{\Lambda_{ij}, 0 \leq i, j \leq n\}$ is a matrix where $\Lambda_{ij} \geq 0$, $i \neq j$ and $\Lambda_i = \sum_{i \neq j} \Lambda_{ij}$. Then exists a unique solution of both Kolmogorov differential equations which satisfies the initial condition $P^0 = \mathbb{I}$. The solution in matrix form is

$$P^t = e^{\Lambda t}$$

where $e^{\Lambda t}$ is an exponential matrix function defined as

$$e^{\Lambda t} = \sum_{k=0}^{\infty} \frac{\Lambda^k t^k}{k!}$$
Example - Kolmogorov Differential Equations

Suppose that $E$ has $p = 2$ states $Y_1, Y_2$ and intensities $\Lambda(Y_1), \Lambda(Y_2)$ are not zero. Then $\Lambda$ has eigenvalues $0$ and $\Lambda = -\Lambda(Y_1) - \Lambda(Y_2)$ with corresponding right eigenvectors $(1, 1)^T, (\Lambda(Y_1), -\Lambda(Y_2))$. Hence

$$\Lambda = \begin{pmatrix} -\Lambda(Y_1) & \Lambda(Y_1) \\ \Lambda(Y_2) & -\Lambda(Y_2) \end{pmatrix} = B \begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix} B^{-1}, \quad B = \begin{pmatrix} 1 & \Lambda(Y_1) \\ 1 & -\Lambda(Y_2) \end{pmatrix}$$

using eigendecomposition.

$$P^t = e^{\Lambda t} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B \begin{pmatrix} 0 & 0 \\ 0 & \Lambda^n \end{pmatrix} B^{-1} = B \begin{pmatrix} 0 & 0 \\ 0 & e^{\Lambda t} \end{pmatrix} B^{-1}$$

$$= \frac{1}{\Lambda(Y_1) + \Lambda(Y_2)} \begin{pmatrix} \Lambda(Y_2) + \Lambda(Y_1)e^{\Lambda t} & \Lambda(Y_1) - \Lambda(Y_1)e^{\Lambda t} \\ \Lambda(Y_2) - \Lambda(Y_2)e^{\Lambda t} & \Lambda(Y_1) + \Lambda(Y_2)e^{\Lambda t} \end{pmatrix}.$$
Forward and backward equations are quite limited utility even in cases when state space is finite (for eg. complex eigenvalues). One common application is to look for a stationary distribution.

**Definition**

Let \( \{X_t, t \geq 0\} \) be Markov jump process with transition matrix \( P^t \).

If vector \( \pi = \{\pi_j, j \geq 0\} \) satisfies

\[
\pi^T = \pi^T P^t
\]

is called *stationary measure* of the process \( \{X_t, t \geq 0\} \) on \( E \) due to \( \{P^t, t \geq 0\} \). If \( \pi \) is also probability distribution on \( E \) then it is called *stationary distribution*. 
Classification of States

Let $\Lambda^* = \{ \Lambda^*_ij, i, j \in E \}$ be matrix with

$$\Lambda^*_ij = \begin{cases} \frac{\Lambda_{ij}}{\Lambda_i}, & \Lambda_i > 0 \\ 0, & \Lambda_i = 0 \end{cases}, i \neq j$$

$$\Lambda^*_{ii} = \begin{cases} 0, & \Lambda_i > 0 \\ 1, & \Lambda_i = 0 \end{cases}$$

and we know that jump times of $\{X_t, t \geq 0\}$ are given by the sequence $S_0, S_1, \ldots$. We define

$$Z_0 = X_0, Z_n = X_{S_n}, n = 1, 2, \ldots$$

It can be shown that $\{Z_n, n \in \mathbb{N}_0\}$ is discrete Markov chain with transition probabilities $\Lambda^*_ij$ defined above.
Classification of States

Theorem

The following properties are equivalent

(i) \( \{Z_n, n \in \mathbb{N}_0\} \) is irreducible,
(ii) for any \( i, j \in E \) we have \( p_{ij}^t > 0 \) for some \( t > 0 \)
(iii) for any \( i, j \in E \) we have \( p_{ij}^t > 0 \) for all \( t > 0 \).

We define \( \{X_t, t \geq 0\} \) to be irreducible if one of the properties (i)-(iii) hold. Similarly it is seen that we can define \( i \) to be transient (reccurent) for \( \{X_t, t \geq 0\} \) if either (i) the set \( \{t : X_t = i\} \) is bounded (unbounded) \( \mathbb{P}_i \)-a.s. (ii) \( i \) is transient (reccurent) for \( \{Z_n, n \in \mathbb{N}_0\} \) or (iii) \( \mathbb{P}_i(\inf\{t > 0, X_t = i, \lim_{s \to t} X_s \neq i\}) < 1(= 1) \).
Suppose that \( \{X_t, t \geq 0\} \) is irreducible and recurrent on \( E \). Then there exists one, and up to a multiplicative factor only one, invariant measure \( \pi \). This has the property \( \pi_j > 0 \) for all \( j \) and can be found in either of the following ways:

(i) for some fixed but arbitrary state \( i \), \( \pi_j \) is the expected time spent in \( j \) between successive entrances to \( i \). That is, with \( \omega(i) = \inf\{t > 0, X_t = i, \lim_{s \uparrow t} X_s \neq i\} \)

\[
\pi_j = \mathbb{E}_i \int_0^{\omega(i)} \mathbb{I}(X_t = j) \, dt
\]

(ii) \( \pi_j = \mu_j / \Lambda_j \) where \( \mu \) is stationary for \( \{Z_n\} \)

(iii) as a solution of \( \pi \Lambda = 0 \).
Ergodicity

An irreducible recurrent process with the stationary measure having finite mass is called \textit{ergodic}, and

\textbf{Theorem}

An irreducible nonexplosive Markov jump process is ergodic if and only if exists a probability solution \( \pi \) \( (\sum_{i \in E} \pi_i = 1, \pi_i \in [0,1]) \) to \( \pi \Lambda = 0 \). In that case \( \pi \) is the stationary distribution.

\textbf{Theorem}

If \( \{X_t, t \geq 0\} \) is ergodic and \( \pi \) is the stationary distribution, then \( p_{ij}^t \to \pi_j, t \to \infty \) for all states \( i, j \in E \).
Ergodicity

As in discrete time, time-average properties like

$$\frac{1}{T} \int_0^T f(X_t) \, dt \xrightarrow{a.s.} \pi(f) = E_\pi(f(X_t)) = \sum_{i \in E} \pi_i f(i)$$

hold under suitable conditions on $f$. It means that the time average converges to the spatial average if the process is ergodic.

**Corollary**

If $\{X_t, t \geq 0\}$ is irreducible recurrent but not ergodic (stationary measure does not have finite mass), then $p^t_{ij} \to 0$ for all states $i, j \in E$. 
Time Reversibility

Time reversibility (or just reversibility) of a process means loosely that the process evolves in just the same way irrespective of whether time is read forward (as usual) or backward.

Definition (Reversibility)

Process \( \{X_t, t \in \mathbb{R}\} \) is time reversible if for finite dimensional distributions and for all \( t \) is

\[
(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \overset{D}{=} (X_{t-t_1}, X_{t-t_2}, \ldots, X_{t-t_n}),
\]

Lemma

If the process \( \{X_t, t \in \mathbb{R}\} \) is time reversible then it is also time stationary.
**Theorem**

Stationary Markov jump process \( \{X_t, t \in \mathbb{R}\} \) with intensity matrix \( \Lambda \) is time reversible if and only if exists probability distribution \( \pi \) on \( E \) satisfying

\[
\pi(x)\Lambda(x, y) = \pi(y)\Lambda(y, x), \quad x, y \in E.
\]

In this case \( \pi \) is stationary distribution.

Previous theorem is also called **detailed balance condition** because the flow between every two states is in balance. The term \( \pi(x)\Lambda(x, y) \) is the probability flow from \( x \) to \( y \).
In contrast to detailed balance condition the equilibrium equation $\pi \Lambda = 0$ gives the **condition of full balance**. More precisely, rewriting $\pi \Lambda = 0$

$$\sum_{x \neq y} \pi(x) \Lambda(x, y) = \sum_{x \neq y} \pi(y) \Lambda(y, x)$$

for all states.

This condition loosely means that everything that flows into some state also flows out of it. Left hand side is the inflow of the state $x$ and right hand side is the outflow.


Thank you for your attention!

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