Linear programming – simplex algorithm, duality and dual simplex algorithm

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Computational Aspects of Optimization
1. Linear programming
2. Primal simplex algorithm
3. Duality in linear programming
4. Dual simplex algorithm
5. Software tools for LP
Linear programming

Standard form LP

\[ \min \ c^T x \]
\[ \text{s.t. } Ax = b, \quad x \geq 0. \]

\[ A \in \mathbb{R}^{m \times n}, \ h(A) = h(A|b) = m. \]

\[ M = \{ x \in \mathbb{R}^n : \ Ax = b, \ x \geq 0 \}. \]
Decomposition of $M$:

- **Convex polyhedron** $P$ – uniquely determined by its vertices (convex hull)
- **Convex polyhedral cone** $K$ – generated by extreme directions (positive hull)

**Direct method** (evaluate all vertices and extreme directions, compute the values of the objective function ...)
One of these cases is valid:

1. $M = \emptyset$
2. $M \neq \emptyset$: the problem is unbounded
3. $M \neq \emptyset$: the problem has an optimal solution (at least one of the solutions is vertex)
Primal simplex algorithm

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**Basis** $B = \text{regular square submatrix of } A$, i.e.

$$A = (B|N).$$

We also consider $B = \{i_1, \ldots, i_m\}$.

We split the objective coefficients and the decision vector accordingly:

$$c^T = (c_B^T, c_N^T),$$

$$x^T(B) = (x_B^T(B), x_N^T(B)),$$

where

$$B \cdot x_B(B) = b, \ x_N(B) \equiv 0.$$

- Feasible basis, optimal basis.
- Basic solution(s).
### Simplex algorithm – simplex table

<table>
<thead>
<tr>
<th></th>
<th>$c^T$</th>
<th>(x^T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_B$</td>
<td>$x_B(B)$</td>
<td>$B^{-1}b$</td>
</tr>
<tr>
<td></td>
<td>(c_B^T B^{-1}b)</td>
<td>(c_B^T B^{-1}A - c^T)</td>
</tr>
</tbody>
</table>
Feasibility condition:

\[ B^{-1}b \geq 0. \]

Optimality condition:

\[ c_B^T B^{-1} A - c^T \leq 0. \]
If the optimality condition is not fulfilled:

- Denote the criterion row by
  \[ \delta^T = c_B^T B^{-1} A - c^T. \]

- Find \( \delta_i > 0 \) and denote the corresponding column by
  \[ \rho = B^{-1} A_{\bullet,i}, \]

  where \( A_{\bullet,i} \) is the \( i \)-th column of \( A \).

- Minimize the ratios
  \[ \hat{u} = \arg \min \left\{ \frac{x_u(B)}{\rho_u} : \rho_u > 0, \ u \in B \right\}. \]

- Substitute \( x_{\hat{u}} \) by \( x_i \) in the basic variables, i.e. \( \hat{B} = B \setminus \{ \hat{u} \} \cup \{ i \}. \)
Denote by $\hat{B}$ the new basis. Define a direction \[ \Delta_u = -\rho_u, \quad u \in B, \]
\[ \Delta_i = 1, \]
\[ \Delta_j = 0, \quad j \notin B \cup \{i\}. \]

If $\rho \leq 0$ ($\hat{u} = \emptyset$), then the problem is unbounded ($c^T x \to -\infty$). Otherwise, we can move from the current basic solution to another one
\[ x(\hat{B}) = x(B) + t\Delta, \]
where $0 \leq t \leq \frac{x(\hat{B})}{\rho_{\hat{u}}}$. We should prove that the new solution is a feasible basic solution and that the objective value decreases ...
New solution is feasible:

\[ x(\hat{B}) \geq 0, \]
\[ Ax(\hat{B}) = Ax(B) + tA\Delta \]
\[ = Ax(B) - tB\rho + tA_{\bullet,i} \]
\[ = b - tBB^{-1}A_{\bullet,i} + tA_{\bullet,i} = b. \]

Objective value decreases

\[ c^T x(\hat{B}) = c^T x(B) + tc^T \Delta \]
\[ = c^T x(B) - tc_B^T \rho + tc_i \]
\[ = c^T x(B) - t(c_B^T B^{-1}A_{\bullet,i} - c_i) \]
\[ = c^T x(B) - t\delta_i, \]

where \( \delta_i > 0 \) is the element of the criterion row.
If $\rho \leq 0$, then $x(\hat{B})$ is feasible for all $t \geq 0$ and the objective value decreases in the direction $\Delta$.

Otherwise the step length $t$ is bounded by $\frac{x_{\hat{u}}(B)}{\rho_{\hat{u}}}$. In this case, the new basis $\hat{B}$ is regular, because we interchange one unit vector by another one using the column $i$ with $\rho_{\hat{u}} > 0$ element (on the right position).
Simplex algorithm – pivot rules

Rules for selecting the entering variable if there are several possibilities:

- **Largest coefficient** in the objective function
- **Largest decrease** of the objective function
- **Steepest edge** – choose an improving variable whose entering into the basis moves the current basic feasible solution in a direction closest to the direction of the vector $c$

$$\max \frac{c^T(x_{new} - x_{old})}{\|x_{new} - x_{old}\|}.$$  

Computationally the most successful.

- **Blands’s rule** – choose the improving variable with the smallest index, and if there are several possibilities for the leaving variable, also take the one with the smallest index (prevents cycling)

Matoušek and Gärtner (2007).
Simplex algorithm – example

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td>2</td>
<td>-2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>2</td>
<td>-2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>5</td>
<td>-3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Moving in direction $\Delta^T = (0, 1, -1, 2)$, i.e.

$$(0, 2, 0, 5) = (0, 0, 2, 1) + t \cdot (0, 1, -1, 2),$$

where $t = 2$. 
Unbounded in direction $\Delta^T = (1, 2, 0, 3)$. 
Content

1. Linear programming
2. Primal simplex algorithm
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Linear programming duality

Primal problem

\[(P) \quad \text{min } c^T x \]
\[\text{s.t. } Ax \geq b, \quad x \geq 0.\]

and corresponding dual problem

\[(D) \quad \text{max } b^T y \]
\[\text{s.t. } A^T y \leq c, \quad y \geq 0.\]
Duality in linear programming

Linear programming duality

Denote

\[ M = \{ x \in \mathbb{R}^n : Ax \geq b, \ x \geq 0 \}, \]
\[ N = \{ y \in \mathbb{R}^m : A^T y \leq c, \ y \geq 0 \}, \]

Weak duality theorem:

\[ b^T y \leq c^T x, \ \forall x \in M, \forall y \in N. \]

Equality holds if and only if (iff) complementarity slackness conditions are fulfilled:

\[ y^T (Ax - b) = 0, \]
\[ x^T (A^T y - c) = 0. \]
Duality theorem: If $M \neq \emptyset$ and $N \neq \emptyset$, than the problems (P), (D) have optimal solutions.

Strong duality theorem: The problem (P) has an optimal solution if and only if the dual problem (D) has an optimal solution. If one problem has an optimal solution, than the optimal values are equal.
Optimize the production of the following products $V_1$, $V_2$, $V_3$ made from materials $M_1$, $M_2$.

<table>
<thead>
<tr>
<th></th>
<th>$V_1$</th>
<th>$V_2$</th>
<th>$V_3$</th>
<th>Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>54 kg</td>
</tr>
<tr>
<td>$M_2$</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>30 kg</td>
</tr>
<tr>
<td>Gain ($/kg$)</td>
<td>10</td>
<td>15</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
Duality in linear programming

Duality

Primal problem

\begin{align*}
\text{max} & \quad 10x_1 + 15x_2 + 10x_3 \\
\text{s.t.} & \quad x_1 + 2x_3 \leq 54, \\
& \quad 2x_1 + 3x_2 + x_3 \leq 30, \\
& \quad x_1 \geq 0, \\
& \quad x_2 \geq 0, \\
& \quad x_3 \geq 0.
\end{align*}

(Dual problem)

\begin{align*}
\text{min} & \quad 54y_1 + 30y_2 \\
\text{s.t.} & \quad y_1 + 2y_2 \geq 10, \\
& \quad 3y_2 \geq 15, \\
& \quad 2y_1 + y_2 \geq 10, \\
& \quad y_1 \geq 0, \\
& \quad y_2 \geq 0.
\end{align*}
Duality

Optimal solution of (D) $\hat{y} = \left(\frac{5}{2}, 5\right)^T$.
Using the complementarity slackness conditions $\hat{x} = (0, 1, 27)^T$.
The optimal values (gains) of (P) and (D) are 285.

- Both (P) constraints are fulfilled with equality, thus there in no material left.
- Dual variables are called **shadow prices** and represent the prices of sources (materials).
- **Sensitivity**: If we increase (P) r.h.s. by one, then the objective value increases by the shadow price.
- The first constraint of (D) is fulfilled with strict inequality with the difference 2.5 $, called **reduced prices**, and the first product is not produced. The producer should increase the gain from $V_1$ by this amount to become profitable.
Transportation problem

- $x_{ij}$ – decision variable: amount transported from $i$ to $j$
- $c_{ij}$ – costs for transported unit
- $a_i$ – capacity
- $b_j$ – demand

ASS. $\sum_{i=1}^{n} a_i \geq \sum_{j=1}^{m} b_j$.  
(Sometimes $a_i, b_j \in \mathbb{N}$.)
Transportation problem

Primal problem

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{j=1}^{m} x_{ij} \leq a_i, \quad i = 1, \ldots, n, \\
& \quad \sum_{i=1}^{n} x_{ij} \geq b_j, \quad j = 1, \ldots, m, \\
& \quad x_{ij} \geq 0.
\end{align*}
\]
Transportation problem

Dual problem

\[
\text{max } \sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j v_j \\
\text{s.t. } u_i + v_j \leq c_{ij}, \\
\quad u_i \leq 0, \\
\quad v_j \geq 0.
\]

Interpretation: \(-u_i\) price for buying a unit of goods at \(i\), \(v_j\) price for selling at \(j\).
Transportation problem

Competition between the transportation company (which minimizes the transportation costs) and an “agent” (who maximizes the earnings):

\[
\sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j v_j \leq \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}
\]
Apply KKT optimality conditions to primal LP ... we will see relations with NLP duality.
Content

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**Primal problem** (standard form)

\[
\text{min } c^T x \\
\text{s.t. } Ax = b, \\
x \geq 0.
\]

and corresponding **dual problem**

\[
\text{max } b^T y \\
\text{s.t. } A^T y \leq c, \\
y \in \mathbb{R}^m.
\]
Dual simplex algorithm works with

- **dual feasible basis** $B$ and
- **basic dual solution** $y(B)$,

where

$$B^T y(B) = c_B,$$

$$N^T y(B) \leq c_N.$$
Primal feasibility $B^{-1}b \geq 0$ is violated until reaching the optimal solution.

Primal optimality condition is always fulfilled:

$$c_B^T B^{-1} A - c^T \leq 0.$$ 

Using $A = (B|N)$, $c^T = (c_B^T, c_N^T)$, we have

$$c_B^T B^{-1} B - c_B^T = 0,$$
$$c_B^T B^{-1} N - c_N^T \leq 0,$$

Setting $\hat{y} = (B^{-1})^T c_B$

$$B^T \hat{y} = c_B,$$
$$N^T \hat{y} \leq c_N.$$ 

Thus, $\hat{y}$ is a basic dual solution.
Dual simplex algorithm – a step

... uses the same simplex table.

- Find index $u \in B$ such that $x_u(B) < 0$ and denote the corresponding row by

$$\tau^T = (B^{-1}A)_u.$$ 

- Denote the criterion row by

$$\delta^T = c_B^T B^{-1} A - c^T \leq 0.$$ 

- Minimize the ratios

$$\hat{i} = \arg \min \left\{ \frac{\delta_i}{\tau_i} : \tau_i < 0 \right\}.$$ 

If there is no $i$ such that $\tau_i < 0$, then STOP: the dual problem is unbounded and primal is infeasible.

- Substitute $x_u$ by $x_{\hat{i}}$ in the basic variables, i.e. $\hat{B} = B \setminus \{u\} \cup \{\hat{i}\}$. We move to another basic dual solution.
The problem is **dual nondegenerate** if for all dual feasible basis $B$ it holds

$$
(A^T y(B) - c)_j = 0, \ j \in B,
$$

$$
(A^T y(B) - c)_j < 0, \ j \notin B.
$$

If the problem is dual nondegenerate, then the dual simplex algorithm ends after finitely many steps.
A general step in the dual simplex algorithm

\[ y(\hat{B}) = y(B) - t(B^{-1})^T \]

with

\[ t := \frac{\delta_i}{\tau_i}. \]

Then it can be shown that the **dual feasibility** is preserved, i.e.

\[ A^T y(\hat{B}) = A^T y(B) - tA^T (B^{-1})^T \leq c, \]

e.g.

\[ (A^T y(\hat{B}))_i = \delta_i + c_i - \frac{\delta_i}{\tau_i} \tau_i = c_i, \]

or

\[ (A^T y(\hat{B}))_u = \delta_u + c_u - \frac{\delta_i}{\tau_i} \tau_u \leq c_u. \]
A general step in the dual simplex algorithm

\[ y(\hat{B}) = y(B) - t(B^{-1})_{\bullet,u}^T \]

with

\[ t := \frac{\delta_i}{\tau_i} > 0. \]

Then it can be shown that the **objective function increases** if the problem is dual nondegenerate, i.e.

\[
\begin{align*}
    b^T y(\hat{B}) &= b^T y(B) - tb^T (B^{-1})_{\bullet,u}^T, \\
    &= b^T y(B) - \frac{\delta_i}{\tau_i} x_u(B) > b^T y(B),
\end{align*}
\]

because \( x_u(B) < 0 \).
Example – dual simplex algorithm

\[ \begin{align*}
\text{min } & 4x_1 + 5x_2 \\
& x_1 + 4x_2 \geq 5, \\
& 3x_1 + 2x_2 \geq 7, \\
& x_1, x_2 \geq 0.
\end{align*} \]
Dual problem

\[
\begin{align*}
\text{max} & \quad -5y_1 - 7y_2 \\
\text{s.t.} & \quad -y_1 - 3y_2 \leq 4 \\
& \quad -4y_1 - 2y_2 \leq 5 \\
& \quad y_1 \leq 0 \\
& \quad y_2 \leq 0.
\end{align*}
\]
Example – dual simplex algorithm

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-5</td>
<td>-1</td>
<td>-4</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>-7</td>
<td>-3</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-4</td>
<td>-5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-8/3</td>
<td>0</td>
<td>-10/3</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>7/3</td>
<td>1</td>
<td>2/3</td>
<td>0</td>
</tr>
<tr>
<td>28/3</td>
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<td>-7/3</td>
<td>0</td>
<td>-4/3</td>
</tr>
<tr>
<td>5</td>
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<td>0</td>
<td>1</td>
<td>-3/10</td>
</tr>
<tr>
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<td>18/10</td>
<td>1</td>
<td>0</td>
<td>2/10</td>
</tr>
<tr>
<td>112/10</td>
<td>0</td>
<td>0</td>
<td>-7/10</td>
<td>-11/10</td>
</tr>
</tbody>
</table>

The last solution is primal and dual feasible, thus optimal.
A general step in the dual simplex algorithm

\[ y(\hat{B}) = y(B) - t(B^{-1})^T \]

i.e.

\[ (0, -4/3) = (0, 0) - 4/3(0, 1), \]

which can be seen in the criterion row in the columns corresponding to the initial basis. Dual constraints 1 and 3 are then active.
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Software tools for LP

- Matlab
- Mathematica
- GAMS
- Cplex studio
- AIMMS
- ...
- R
- MS Excel
- ...

...

