**Motivation and applications**

**Knapsack problem**

Values $a_1 = 4$, $a_2 = 6$, $a_3 = 7$, costs $c_1 = 4$, $c_2 = 5$, $c_3 = 11$, budget $b = 10$:

$$\text{max } 3 \sum_{i=1}^{3} c_i x_i$$

$$\text{s.t. } 3 \sum_{i=1}^{3} a_i x_i \leq 10,$$

$$x_i \in \{0, 1\}.$$

Consider $=$ instead of $\leq$, $0 \leq x_i \leq 1$ and rounding instead of $x_i \in \{0, 1\}$, heuristic (ratio $c_i/a_i$) ...
Scheduling to Minimize the Makespan

- $i$ machines, $j$ jobs,
- $y$ - machine makespan,
- $x_{ij}$ - assignment variable,
- $t_{ij}$ - time necessary to process job $j$ on machine $i$.

\[
\min \ y \\
\text{s.t. } \sum_{j=1}^{m} x_{ij} = 1, \ j = 1, \ldots, n, \\
\sum_{j=1}^{m} t_{ij} x_{ij} \leq y, \ i = 1, \ldots, m, \\
x_{ij} \in \{0, 1\}, \ y \geq 0.
\]

Lot Sizing Problem

- $x_t$ - production at period $t$,
- $y_t$ - on/off decision at period $t$,
- $s_t$ - inventory at the end of period $t$ ($s_0$ fixed),
- $D_t$ - (predicted) expected demand at period $t$,
- $p_i$ - unit production costs at period $t$,
- $f_i$ - setup costs at period $t$,
- $h_t$ - inventory costs at period $t$,
- $M$ - a large constant.

\[
\min \ \sum_{t=1}^{T} (p_i x_t + f_i y_t + h_t s_t) \\
\text{s.t. } s_{t-1} + x_t - D_t = s_t, \ t = 1, \ldots, T, \\
x_t \leq M y_t, \\
x_t, s_t \geq 0, \ y_t \in [0, 1].
\]

ASS. Wagner-Whitin costs $p_{t+1} \leq p_t + h_t$.

Unit Commitment Problem

- $i = 1, \ldots, n$ units (power plants), $t = 1, \ldots, T$ periods,
- $y_{it}$ - on/off decision for unit $i$ at period $t$,
- $x_{it}$ - production level for unit $i$ at period $t$,
- $D_t$ - (predicted) expected demand at period $t$,
- $P_{min}, P_{max}$ - minimal/maximal production capacity of unit $i$,
- $c_i$ - variable production costs,
- $f_i$ - (fixed) start-up costs.

\[
\min \ \sum_{i=1}^{n} \sum_{t=1}^{T} (c_i x_{it} + f_i y_{it}) \\
\text{s.t. } \sum_{i=1}^{n} x_{it} \geq D_t, \ t = 1, \ldots, T, \\
P_{min} y_{it} \leq x_{it} \leq P_{max} y_{it}, \\
x_{it} \geq 0, \ y_{it} \in [0, 1].
\]
Sparse $l_1$ regression

- $Y_i$ – dependent variable $i = 1, \ldots, n$
- $X_i$ – explanatory (independent) variables $j = 1, \ldots, m$
- $\beta_j$ – coefficients.

$$\min_{\beta} \sum_{i=1}^{n} \left| Y_i - \sum_{j=1}^{m} X_{ij} \beta_j \right|$$  \hspace{1cm} (5)

s.t. at most $\kappa < m$ coefficients are nonzero.

MILP reformulation

$$\min_{\beta, u, z} \sum_{i=1}^{n} u_i^+ + u_i^-
\text{s.t.} \quad u_i^+ - u_i^- = Y_i - \sum_{j=1}^{m} X_{ij} \beta_j,
- M z_j \leq \beta_j \leq M z_j,
\sum_{j=1}^{m} z_j \leq \kappa,
\begin{align*}
  &u_i^+ \geq 0, \quad u_i^- \geq 0, \quad z_j \in \{0, 1\},
\end{align*}$$  \hspace{1cm} (6)

Chance constrained problems – single random constraint

Let $f, g(\cdot, \xi) : \mathbb{R}^n \to \mathbb{R}$ be real functions, $X \subseteq \mathbb{R}^n$, $\xi$ be a real random vector, $\varepsilon \in (0, 1)$ small:

$$\min_{x \in X} f(x)$$

s.t. $P(g(x, \xi) \leq 0) \geq 1 - \varepsilon.$

INTERPRETATION: for a given $x \in X$, the probability of $\xi$ for which the random constraint is fulfilled must be at least $1 - \varepsilon$:

$$P(g(x, \xi) \leq 0) = P(\{\xi : g(x, \xi) \leq 0\}).$$

Example: Value at Risk (VaR).

Integer linear programming

$$\min c^T x$$  \hspace{1cm} (8)

$$Ax \geq b,$$  \hspace{1cm} (9)

$$x \in \mathbb{Z}_+^n.$$  \hspace{1cm} (10)

Assumption: all coefficients are integer (rational before multiplying by a proper constant).

Set of feasible solution and its relaxation

$$S = \{x \in \mathbb{Z}_+^n : A x \geq b\},$$  \hspace{1cm} (11)

$$P = \{x \in \mathbb{R}_+^n : A x \geq b\}.$$  \hspace{1cm} (12)

Obviously $S \subseteq P$. Not so trivial that $S \subseteq \text{conv}(S) \subseteq P.$
ILP – irrational data

Škoda (2010):

\[
\begin{align*}
\text{max } & \sqrt{2}x - y \\
\text{s.t. } & \sqrt{2}x - y \leq 0, \\
& x \geq 1, \\
& x, y \in \mathbb{N}.
\end{align*}
\]

(13)

The objective value is bounded (from above), but there is no optimal solution.

For any feasible solution with the objective value \(z = \sqrt{2}x^* - \lfloor \sqrt{2}x^* \rfloor\) we can construct a solution with a higher objective value...

ILP – irrational data

Let \(z = \sqrt{2}x^* - \lfloor \sqrt{2}x^* \rfloor\) be the optimal solution. Since \(-1 < z < 0\), we can find \(k \in \mathbb{N}\) such that \(kz < -1\) and \((k-1)z > -1\). By setting \(\epsilon = -1 - kz\) we get that 

\[
\begin{align*}
\sqrt{2}kx^* - \lfloor \sqrt{2}kx^* \rfloor & = kz + k \lfloor \sqrt{2}x^* \rfloor - \lfloor \sqrt{2}kx^* \rfloor \\
& = -1 - \epsilon + k \lfloor \sqrt{2}x^* \rfloor - \lfloor \sqrt{2}kx^* \rfloor \\
& = k \lfloor \sqrt{2}x^* \rfloor - 1 - \epsilon - \lfloor \sqrt{2}kx^* \rfloor - 1 - \epsilon \\
& = -\epsilon > z.
\end{align*}
\]

(14)

\((k \lfloor \sqrt{2}x^* \rfloor - 1\) is integral)

Thus, we have obtained a solution with a higher objective value which is a contradiction.

Example

Consider set \(S\) given by

\[
\begin{align*}
7x_1 + 2x_2 & \geq 5, \\
7x_1 + x_2 & \leq 28, \\
-4x_1 + 14x_2 & \leq 35, \\
x_1, x_2 & \in \mathbb{Z}_+.
\end{align*}
\]

Set of feasible solutions, its relaxation and convex envelope
Formulation and properties

**Integer linear programming problem**

Problem

\[
\min c^T x : x \in S. \tag{15}
\]

is equivalent to

\[
\min c^T x : x \in \text{conv}(S). \tag{16}
\]

\(\text{conv}(S)\) is very difficult to construct – many constraints ("strong cuts") are necessary (there are some important exceptions).

LP-relaxation:

\[
\min c^T x : x \in P. \tag{17}
\]

**Mixed-integer linear programming**

Often both integer and continuous decision variables appear:

\[
\min c^T x + d^T y \\
\text{s.t.} \ Ax + By \geq b \\
x \in \mathbb{Z}^n, \ y \in \mathbb{R}^m.
\]

(WE DO NOT CONSIDER IN INTRODUCTION)

Formulation and properties

**Basic algorithms**

We consider:
- Cutting Plane Method
- Branch-and-Bound

There are methods which combine the previous alg., e.g. Branch-and-Cut (add cuts to reduce the problem for B&B).

Formulation and properties

**Cutting plane method – Gomory cuts**

1. Solve LP-relaxation using (primal or dual) SIMPLEX algorithm.
   - If the solution is integral – END, we have found an optimal solution,
   - otherwise continue with the next step.
2. Add a Gomory cut (...) and solve the resulting problem using DUAL SIMPLEX alg.
Cutting plane method

Example

\[
\begin{align*}
\min 4x_1 + 5x_2 & \quad (18) \\
x_1 + 4x_2 & \geq 5, \quad (19) \\
3x_1 + 2x_2 & \geq 7, \quad (20) \\
x_1, x_2 & \in \mathbb{Z}_+. \quad (21)
\end{align*}
\]

Dual simplex for LP-relaxation ...

After two iterations of the dual SIMPLEX algorithm ...

<table>
<thead>
<tr>
<th></th>
<th>4</th>
<th>5</th>
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<th>0</th>
<th>x_1</th>
<th>x_2</th>
<th>x_3</th>
<th>x_4</th>
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<td>1</td>
<td>0</td>
<td>2/10</td>
<td>-4/10</td>
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<td></td>
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<td>0</td>
<td>0</td>
<td>-7/10</td>
<td>-11/10</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Gomory cuts

There is a row in simplex table, which corresponds to a non-integral solution \( x_i \) in the form:

\[
x_i + \sum_{j \in N} w_{ij}x_j = d_i, \quad (22)
\]

where \( N \) denotes the set of non-basic variables; \( d_i \) is non-integral. We denote

\[
w_{ij} = \lfloor w_{ij} \rfloor + f_{ij}, \quad (23)
\]

\[
d_i = \lfloor d_i \rfloor + f_i, \quad (24)
\]

i.e. \( 0 \leq f_{ij}, f_i < 1 \).

\[
\sum_{j \in N} f_{ij}x_j \geq f_i, \quad (25)
\]

or rather \( -\sum_{j \in N} f_{ij}x_j + s = -f_i, \ s \geq 0 \).
Gomory cuts

General properties of cuts (including Gomory ones):

- Property 1: Current (non-integral) solution becomes infeasible (it is cut).
- Property 2: No feasible integral solution becomes infeasible (it is not cut).

We express the constraints in the form

\[ x_i + \sum_{j \in N} \lfloor w_{ij} \rfloor x_j = \lfloor d_i \rfloor + f_i, \]  
(26)

\[ x_i + \sum_{j \in N} [w_{ij}] x_j - \lfloor d_i \rfloor = f_i - \sum_{j \in N} f_{ij} x_j, \]  
(27)

Current solution \( x_i^* = 0 \) for \( j \in N \) and \( x_i^* = d_i \) is non-integral, i.e. \( 0 < x_i^* - \lfloor d_i \rfloor < 1 \), thus

\[ 0 < x_i^* - \lfloor d_i \rfloor = f_i - \sum_{j \in N} f_{ij} x_j^* \]  
(28)

and

\[ \sum_{j \in N} f_{ij} x_j^* < f_i, \]  
(29)

which is a contradiction with the Gomory cut.

Consider an arbitrary integral feasible solution and rewrite the constraint as

\[ x_i + \sum_{j \in N} [w_{ij}] x_j - \lfloor d_i \rfloor = f_i - \sum_{j \in N} f_{ij} x_j, \]  
(30)

Left-hand side (LS) is integral, thus right-hand side (RS) is integral. Moreover, \( f_i < 1 \) a \( \sum_{j \in N} f_{ij} x_j \geq 0 \), thus RS is strictly lower than 1 and at the same time it is integral, thus lower or equal to 0, i.e. we obtain Gomory cut

\[ f_i - \sum_{j \in N} f_{ij} x_j \leq 0. \]  
(31)

Thus each integral solution fulfills it.
Cutting plane method

Dantzig cuts

\[ \sum_{j \in N} x_j \geq 1. \]  
(32)

(Remind that non-basic variables are equal to zero.)

After two iterations of the dual SIMPLEX algorithm...

<table>
<thead>
<tr>
<th></th>
<th>4 ( x_1 )</th>
<th>5 ( x_2 )</th>
<th>0 ( x_3 )</th>
<th>0 ( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>1</td>
<td>-3/10</td>
</tr>
<tr>
<td>4 ( x_1 )</td>
<td>18/10</td>
<td>1</td>
<td>0</td>
<td>2/10</td>
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<tr>
<td>112/10</td>
<td>0</td>
<td>0</td>
<td>-7/10</td>
<td>-11/10</td>
</tr>
</tbody>
</table>

For example, \( x_1 \) is not integral:

\[ x_1 + \frac{2}{10}x_3 - \frac{4}{10}x_4 = \frac{18}{10}, \]
\[ x_1 + (0 + \frac{2}{10})x_3 + (-1 + \frac{6}{10})x_4 = 1 + \frac{8}{10}. \]

Gomory cut:

\[ \frac{2}{10}x_3 + \frac{6}{10}x_4 \geq \frac{8}{10}. \]

New simplex table

<table>
<thead>
<tr>
<th></th>
<th>4 ( x_1 )</th>
<th>5 ( x_2 )</th>
<th>0 ( x_3 )</th>
<th>0 ( x_4 )</th>
<th>0 ( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 ( x_2 )</td>
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</tr>
<tr>
<td>4 ( x_1 )</td>
<td>18/10</td>
<td>1</td>
<td>0</td>
<td>2/10</td>
<td>-4/10</td>
</tr>
<tr>
<td>0 ( x_5 )</td>
<td>-8/10</td>
<td>0</td>
<td>0</td>
<td>-2/10</td>
<td>-6/10</td>
</tr>
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<td>-11/10</td>
<td>0</td>
</tr>
</tbody>
</table>

Dual simplex alg. ... Gomory cut:

\[ \frac{4}{6}x_3 + \frac{1}{6}x_5 \geq \frac{2}{3}. \]

Dual simplex alg. ... optimal solution (2, 1, 1, 1, 0, 0).