4 Multiple Life Insurance

Please read the theory in Chapter 8 of Gerber book. You can\(^1\) also solve the examples in Appendix C.8.

We consider \(m\) independent\(^2\) lives with (random) future lifetimes

\[
T_1 := T_{x_1}, \ldots, T_m := T_{x_m}.
\]

Notation

- Joint-life status
  - status \(u = x_1 : x_2 : \cdots : x_m\) = all \(m\) participating lives survive,
  - failure time
    \[
    T(u) = \min\{T_1, \ldots, T_m\},
    \]
  - survival probability
    \[
    tp_{x_1:x_2: \cdots: x_m} = P(T(u) > t) = \prod_{k=1}^{m} P(T_k > t) = \prod_{k=1}^{m} tp_{x_k},
    \]
    and
    \[
    tq_{x_1:x_2: \cdots: x_m} = 1 - tp_{x_1:x_2: \cdots: x_m}.
    \]

Under independence, we set

\[
l_{x_1:x_2: \cdots: x_m} = \prod_{k=1}^{m} l_{x_k},
\]

\[
d_{x_1:x_2: \cdots: x_m} = l_{x_1:x_2: \cdots: x_m} - l_{x_1+1:x_2+1: \cdots: x_m+1}.
\]

Then

\[
sp_{x_1:x_2: \cdots: x_m} = \frac{l_{x_1+t:x_2+t: \cdots: x_m+t}}{l_{x_1:x_2: \cdots: x_m}}.
\]

- Last-survivor status
  - status \(u = \overline{x_1 : x_2 : \cdots : x_m}\) = at least one of the \(m\) lives survives,
  - failure time
    \[
    T(u) = \max\{T_1, \ldots, T_m\},
    \]

\(^1\)It is not mandatory, but it can help you.
\(^2\)The independence is quite questionable assumption, especially when we consider a family insurance. There are several approaches how to elaborate the dependence, e.g., copula functions or conditional forces of mortality.
survival probability

\[ t_P_{x_1,x_2,\ldots,x_m} = P(T(u) > t) = S_1^t - S_2^t + \cdots (-1)^{m-1} S_m^t, \]

where

\[ S_k^t = \sum_{(j_1, \ldots, j_k) \subset \{1, \ldots, m\}} t_P_{x_1,j_2,\ldots,j_k}, \]

and

\[ t_Q_{x_1,x_2,\ldots,x_m} = 1 - t_P_{x_1,x_2,\ldots,x_m}. \]

**Example 4.1** Consider the following insurances for a pair of independent lifes at ages \( x \) and \( y \):

- **a)** joint-life whole life insurance payable on the first death,
- **b)** joint-life life annuity-due.
- **c)** joint-life life annuity-due for \( n \) years.

Derive a reasonable generalization of the commutation functions which enable you to simplify the computation of the net single premiums.

**Solution:** a) Under the independence of lifes

\[
A_{x:y} = \sum_{k=0}^{\infty} v^{k+1} \cdot k \cdot P_{x:y} q_{x:y}
\]

\[
= \sum_{k=0}^{\infty} v^k \cdot l_{x:y} \cdot d_{x+k:y+k} \cdot l_{x+k:y+k}
\]

\[
= \sum_{k=0}^{\infty} \frac{v^k f(x+y+k)}{v^k f(x,y)} \cdot \frac{d_{x+k:y+k}}{l_{x:y}}.
\]

There are several possible choices of \( f \):

\[ f(x, y) = \frac{x + y}{2}, \quad f(x, y) = \max\{x, y\}, \quad f(x, y) = \min\{x, y\}. \]

On the other hand, it is not possible to use simple sum of the ages as \( f \), because we need a transformation which preserves \( v^k \). So, if we define the commutation functions as

\[
C_{x:y} = v^f(x+1,y+1) \cdot d_{x:y}, \quad D_{x:y} = v^f(x,y) \cdot l_{x:y},
\]

\[
M_{x:y} = \sum_{k=0}^{\infty} C_{x+k:y+k}, \quad N_{x:y} = \sum_{k=0}^{\infty} D_{x+k:y+k},
\]

\[
R_{x:y} = \sum_{k=0}^{\infty} M_{x+k:y+k}, \quad S_{x:y} = \sum_{k=0}^{\infty} N_{x+k:y+k},
\]
we can get the standard expression for NSP

\[ A_{x:y} = \sum_{k=0}^{\infty} \frac{C_{x+k:y+k}}{D_{x:y}} = M_{x:y} - D_{x:y}. \]

b) For the life annuity-due, we obtain

\[
\ddot{a}_{x:y} = \sum_{k=0}^{\infty} v^k k p_{x:y} = \sum_{k=0}^{\infty} v^k k l_{x+k:y+k} l_{x:y} = \sum_{k=0}^{\infty} v f(x+k+1,y+k+1) l_{x+k:y+k} l_{x:y} = \sum_{k=0}^{\infty} \frac{D_{x+k:y+k}}{D_{x:y}} = N_{x:y}. 
\]

c) For the life annuity-due for \( n \) years, we have

\[
\ddot{a}_{x:y} = \sum_{k=0}^{n-1} v^k k p_{x:y} = \sum_{k=0}^{n-1} \frac{D_{x+k:y+k}}{D_{x:y}} = N_{x:y} - N_{x+n:y+n}. 
\]

\[ \square \]

**Remark 4.2** The generalization of the commutation functions to \( m \) lifes is straightforward, e.g.

\[ C_{x_1:x_2:...:x_m} = v f(x_1+1,...,x_m+1) d_{x_1:x_2:...:x_m}, \quad D_{x_1:x_2:...:x_m} = v f(x_1,...,x_m) l_{x_1:x_2:...:x_m}, \]

where

\[ f(x_1,\ldots,x_m) = \frac{\sum_{k=1}^{m} x_k}{m}, \text{ or } f(x_1,\ldots,x_m) = \max\{x_1,\ldots,x_m\}, \text{ or } f(x_1,\ldots,x_m) = \min\{x_1,\ldots,x_m\}. \]

Note that also the relations between CF which we know from the univariate case are valid, e.g.

\[ M_{x:y} = D_{x:y} - d N_{x:y}. \]

**Example 4.3** Verify that

\[ A_{x:y} = 1 - d \cdot \ddot{a}_{x:y}, \quad A_{\overline{x:y}} = 1 - d \cdot \ddot{a}_{\overline{x:y}}. \]
Solution: One possibility is to use the following identity
\[ 1 + v + \cdots + v^K = \frac{1 - v^{K+1}}{1 - v}, \]
and compute the expected value with respect to one of the following distribution of curtate future lifetime \( K \):
\[
P(K = k) = k p_x q_{xy}, \quad \text{or} \quad P(K = k) = k p_{x+y} q_{x+y}.
\]
Other possibilities are a direct derivation using the formula for the net single premiums, or using the relations between the commutation functions. □

Example 4.4 Consider the following insurances for a pair of independent lifes at ages \( x \) and \( y \):

a) last-survival life annuity-due.

b) last-survival whole life insurance payable on the last death,

Using the above introduced commutation functions derive the net single premiums.

Solution: a)
\[
\bar{\alpha}_{x:y} = \sum_{k=0}^{\infty} v^k k p_{x:y} \\
= \sum_{k=0}^{\infty} v^k (k p_x + k p_y - k p_{x:y}) \\
= \sum_{k=0}^{\infty} v^k k p_x + \sum_{k=0}^{\infty} v^k k p_y - \sum_{k=0}^{\infty} v^k k p_{x:y} \\
= \frac{N_x}{D_x} + \frac{N_y}{D_y} - \frac{N_{x:y}}{D_{x:y}}.
\]
b) We can use the previous example to get
\[
\bar{A}_{x:y} = 1 - d \bar{\alpha}_{x:y} \\
= 1 + 1 - 1 - d \left( \frac{N_x}{D_x} + \frac{N_y}{D_y} - \frac{N_{x:y}}{D_{x:y}} \right) \\
= \frac{D_x - d N_x}{D_x} + \frac{D_y - d N_y}{D_y} - \frac{D_{x:y} - d N_{x:y}}{D_{x:y}} \\
= \frac{M_x}{D_x} + \frac{M_y}{D_y} - \frac{M_{x:y}}{D_{x:y}}.
\]
□
Example 4.5 Consider

a) widow’s annuity-due (asymmetric) – payment stream of rate 1 starts at the death of husband \(x\) and terminates at the death of wife \(y\).

b) widow’s and widower’s annuity-due (symmetric) – payment stream starts at the death of husband \(x\) or wife \(y\) and terminates at the death of wife \(y\) or husband \(x\).

c) orphan’s annuity-due – payment stream starts at the death of parents \(x, y\) and terminates at the death of child \(z\) or by reaching the age of 18.

Solution: a) Denote by \(u\) the status when wife is living and husband died

\[
kP_u^{(a)} = kpy(1 - kp_x).
\]

Then

\[
\ddot{a}_u^{(a)} = \sum_{k=0}^{\infty} v^k kP_u^{(a)}
= \sum_{k=0}^{\infty} v^k kpy(1 - kp_x)
= \ddot{a}_y - \ddot{a}_{x:y}.
\]

b) Denote by \(u\) the status when the wife is living and the husband died or vice versa

\[
kP_u^{(b)} = kpy(1 - kp_x) + kp_x(1 - kp_y).
\]

Then

\[
\ddot{a}_u^{(b)} = \sum_{k=0}^{\infty} v^k kP_u^{(b)}
= \sum_{k=0}^{\infty} v^k kpy(1 - kp_x) + kp_x(1 - kp_y)
= \ddot{a}_x + \ddot{a}_y - 2\ddot{a}_{x:y}.
\]

c) Denote by \(u\) the status when the child is living and the parents died and set \(n = 18 - z\). Then

\[
kP_u^{(c)} = kp_z(1 - kp_x)(1 - kp_y),
\]

and

\[
\ddot{a}_u^{(c)} = \sum_{k=0}^{n-1} v^k kP_u^{(c)}
= \sum_{k=0}^{n-1} v^k kp_z(1 - kp_x)(1 - kp_y)
= \ddot{a}_z m - \ddot{a}_{x;z} m - \ddot{a}_{y;z} m + \ddot{a}_{x:y;z} m.
\]

\(\square\)
Example 4.6 Consider orphan’s annuity-due where payment stream starts at the death of parents $x, y$ and terminates when both children $z, w$ reach the age of 18 or at the death of last child.