Zero-sum games of two players

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Computational Aspects of Optimization
Zero-sum games of two players

Definition

A triplet \( \{X, Y, K\} \) is called a **game of two rational players with zero sum**, if

1. \( X \) is a set of strategies of Player 1 (P1),
2. \( Y \) is a set of strategies of Player 2 (P2),
3. \( K : X \times Y \to \mathbb{R} \) is a payoff function of player 1, i.e. if P1 plays \( x \in X \) and P2 plays \( y \in Y \), then P1 gets \( K(x, y) \) and P2 gets \(-K(x, y)\).
Zero-sum games of two players

Definition

For the zero-sum games \(X, Y, K\) we define

- **upper value** of the game \(uv^* = \inf_{y \in Y} \sup_{x \in X} K(x, y)\),
- **lower value** of the game \(lv^* = \sup_{x \in X} \inf_{y \in Y} K(x, y)\),
- **upper price** of the game \(up = \min_{y \in Y} \sup_{x \in X} K(x, y)\),
- **lower price** of the game \(lp = \max_{x \in X} \inf_{y \in Y} K(x, y)\).

If the lower and upper prices exist and it holds \(up = lp\), then we say that the game has the **price** \(p = up = lp\).

Upper value can be seen as the lowest payoff of P1, if P1 knows strategy of P2 before his/her move.
Zero-sum games of two players

Definition
We say that
- \( \hat{x} \in X \) is an optimal strategy of P1, if \( K(\hat{x}, y) \geq lv^* \) for all \( y \in Y \).
- \( \hat{y} \in Y \) is an optimal strategy of P2, if \( K(x, \hat{y}) \leq uv^* \) for all \( x \in X \).
Zero-sum games of two players

**Proposition**

*For each zero-sum game* \(\{X, Y, K\} \) *the upper and lower value exits and it holds*

\[
lv^* \leq uv^*.\]

*For each* \(\tilde{x} \in X\) *and* \(\tilde{y} \in Y\) *it holds*

\[
\inf_{y \in Y} K(\tilde{x}, y) \leq K(\tilde{x}, \tilde{y}),
\]

\[
\sup_{x \in X} \inf_{y \in Y} K(x, y) \leq \sup_{x \in X} K(x, \tilde{y}),
\]

\[
lv^* = \sup_{x \in X} \inf_{y \in Y} K(x, y) \leq \inf_{y \in Y} \sup_{x \in X} K(x, y) = uv^*. \tag{1}\]
Zero-sum games of two players

Proposition

For each zero-sum game \( \{X, Y, K\} \) it holds that

- There is at least one optimal strategy of P1, if and only if the lower price exists.
- There is at least one optimal strategy of P2, if and only if the upper price exists.

\[\Rightarrow\]: Let \( \hat{x} \in X \) be an optimal strategy of P1, i.e. \( K(\hat{x}, y) \geq lv^* \) for all \( y \in Y \). Then

\[
lv^* \leq \inf_{y \in Y} K(\hat{x}, y) \leq \sup_{x \in X} \inf_{y \in Y} K(x, y) = lv^*. \tag{2}
\]

Thus

\[
lv^* = \inf_{y \in Y} K(\hat{x}, y) = \max_{x \in X} \inf_{y \in Y} K(x, y) = lp. \tag{3}
\]
Proposition

Let \( \{X, Y, K\} \) be a zero-sum game with \( X, Y \) compact and \( K \) continuous. Then the upper and lower prices exist.
A zero-sum game \( \{X, Y, K\} \) has a price if and only if the payoff function has a saddle point, i.e. there is a pair \( (\hat{x}, \hat{y}) \) such that

\[
K(x, \hat{y}) \leq K(\hat{x}, \hat{y}) \leq K(\hat{x}, y)
\]

for all \( x \in X \) and \( y \in Y \). Then \( \hat{x} \) is an optimal strategy for P1, \( \hat{y} \) is an optimal strategy for P2, and \( p = K(\hat{x}, \hat{y}) \) is the price of the game.

\(^{\text{a}}\)Such pair can be seen as a Nash equilibrium for two player games.

“\( \Rightarrow \)” : \( K(x, \hat{y}) \leq p \leq K(\hat{x}, y) \).
Theorem

Let \( \{X, Y, K\} \) be a zero-sum game where \( X, Y \) are nonempty convex compact sets and \( K(x, y) \) is continuous, concave in \( x \) and convex in \( y \). Then, there exists the price of the game, i.e.

\[
\min_{y \in Y} \max_{x \in X} K(x, y) = \max_{x \in X} \min_{y \in Y} K(x, y).
\]

Applicable also out of the game theory, e.g. in robustness.

Generalizations: Rockafellar (1970)
Matrix games

Definition

We say that $\{X, Y, A\}$ is a **matrix game** if it a zero sum game (of two players), $A \in \mathbb{R}^{n \times m}$ is a matrix, and

$$K(x, y) = x^T Ay,$$

$$X = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, \ x_i \geq 0 \right\},$$

$$Y = \left\{ y \in \mathbb{R}^m : \sum_{j=1}^{m} y_j = 1, \ y_j \geq 0 \right\}.$$
Rock–paper–scissors

\[ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \]
Rock–paper–scissors–lizard–Spock
Rock–paper–scissors–lizard–Spock
Definition

For a matrix game \(\{X, Y, A\}\), we define a matrix game with \textbf{pure strategies} \(\{\overline{X}, \overline{Y}, A\}\), where

\[
\overline{X} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, \; x_i \in \{0, 1\} \right\},
\]

\[
\overline{Y} = \left\{ y \in \mathbb{R}^m : \sum_{j=1}^{m} y_j = 1, \; y_j \in \{0, 1\} \right\}.
\]

We say that \(\{X, Y, A\}\) has a \textbf{price in pure strategies} if both players have optimal pure strategies.
Matrix games

Proposition

*Each matrix game has a price and both players have optimal strategies.*

Proposition

*Matrix game \( \{X, Y, A\} \) has a price in pure strategies if and only if \( \{\overline{X}, \overline{Y}, A\} \) has a price.*
Proposition

Let \( \{X, Y, A\} \) be a matrix game and \( \hat{x} \in X \) and \( \hat{y} \in Y \) with price \( p \). Then

1. \( \hat{x} \) is an optimal strategy of P1 if and only if \( \hat{x}^T A \geq (p, \ldots, p) \),
2. \( \hat{y} \) is an optimal strategy of P2 if and only if \( A\hat{y} \leq (p, \ldots, p)^T \).

\[
\hat{x}^T A \geq (p, \ldots, p) \iff \hat{x}^T Ay \geq p, \ \forall y \in Y.
\]

("\( \Rightarrow \)" \( y \) & \( \sum_i y_i = 1 \), "\( \Leftarrow \)" \( y = e_i \))
Proposition

(Complementarity conditions) Let \(\{X, Y, A\}\) be a matrix game with price \(p\) and let \(\hat{x} \in X\) and \(\hat{y} \in Y\) be optimal strategies. Then

1. if \(\hat{x}_i > 0\), then \((A\hat{y})_i = p\)
2. if \(\hat{y}_j > 0\), then \((\hat{x}^T A)_j = p\).
Consider

\[ A = \begin{pmatrix} 5 & 1 \\ 0 & 7 \end{pmatrix} \]

\[ 5x_1 \geq p, \quad x_1 + 7x_2 \geq p, \quad x_1 + x_2 = 1, \quad x_1 \geq 0, \quad x_2 \geq 0 \]

\[ \max_{x \in X} \min \{5x_1, x_1 + 7x_2\} = p \]

and using \( x_1 + x_2 = 1 \)

\[ \max_{x_1 \geq 0} \min \{5x_1, 7 - 6x_1\} = p \]

Maximum is attained at \( \hat{x}_1 = 7/11, \hat{x}_2 = 4/11 \) with the price \( p = 35/11 \).

Using complementarity conditions, we obtain \( \hat{y}_1 = 6/11, \hat{y}_2 = 5/11 \).
Let $a, b \in \mathbb{R}^n$. We say that $a$ strictly dominates $b$ ($b$ is strictly dominated by $a$), if $a_i > b_i$ for all $i = 1, \ldots, n$.

**Proposition**

Let $\{X, Y, A\}$ be a matrix game.

1. If a row $A_{k,i}$ is strictly dominated by a convex combination of other rows, then each optimal strategy of $P1$ fulfills $\hat{x}_k = 0$.

2. If a column $A_{i,k}$ strictly dominates a convex combination of other columns, then each optimal strategy of $P2$ fulfills $\hat{y}_k = 0$. 
Matrix games

\[
\begin{pmatrix}
3 & 2 & 4 & 0 \\
3 & 4 & 2 & 3 \\
6 & 5 & 5 & 1 \\
1 & 4 & 0 & 7 \\
\end{pmatrix}
\]

Show that \((0, 0, 7/11, 4/11)^T\) is optimal strategy for P1, 
\((0, 0, 6/11, 5/11)^T\) for P2, and the price is \(p = 35/11\).
Matrix games

Proposition

Matrix game \( \{X, Y, A\} \) has a price \( p \) in pure strategies if and only if matrix \( A \) has a saddle point, i.e. there is a pair of indices \( \{k, l\} \) such that

\[
A_{kl} = \min\{A_{kj} : j = 1, \ldots, m\} = \max\{A_{il} : i = 1, \ldots, n\}.
\]

(minimum in the row, maximum in the column)

e_k, e_l are optimal strategies of P1, P2

\[
\iff \\
(e_k^T A)_j = A_{kj} \geq p, \forall j, \\
(A e_l)_i = A_{il} \leq p, \forall i, \\
\]

\[
\iff \\
A_{kl} = \min\{A_{kj} : j = 1, \ldots, m\} = \max\{A_{il} : i = 1, \ldots, n\}.
\]
Find the saddle point(s) ..

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{pmatrix}, \quad \begin{pmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
3 & 2 & 2 \\
\end{pmatrix},
\]

(8)
