Diversification-consistent DEA-risk tests – solution techniques and an empirical comparison

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Abstract.
In this paper, we will propose numerically tractable formulations of the diversification-consistent DEA tests, which generalize traditional DEA tests as well as mean-risk models. We employ general deviation measures to measure risk of the investment opportunities. We will compare strength of the tests and give characterizations of efficient and inefficient investment opportunities. US industry representative portfolios will be ranked using the proposed DEA tests.

Keywords: Data envelopment analysis, diversification, deviation measures, efficiency, US industry representative portfolios

JEL classification: C44
AMS classification: 90C15

1 Introduction

Data Envelopment Analysis (DEA) was introduced by Charnes, Cooper and Rhodes [11] as a tool for testing efficiency of decision making units with the same structure of consumed inputs and produced outputs. Many generalizations have been proposed since then taking into account various aspects of production theory and practical applications, see, e.g., Cooper et al. [13]. A special attention has been given to applications of DEA models in finance. Murthy et al. [19] accessed in their seminal work the efficiency of mutual funds based on their mean-risk profile and transaction costs. More precisely, the indicators, which are preferably minimized by investors (e.g. risk, transaction costs), served as the inputs, and those, which are maximized (e.g. expected return), were used as the outputs. DEA model with variable return to scale introduced by Banker et al. [1] was employed. Similar efficiency tests based on DEA were proposed also by Basso and Funari [2], Branda and Kopa [8], Chen and Lin [12], where also other inputs and outputs were considered.

Several authors realized that the standard DEA tests employed in finance do not take into account dependencies between the assets. Namely, risk measures used as the inputs were combined linearly in the dual formulations, which does not correspond to proper diversification. It was even shown by Branda [6] that ignoring diversification leads to weaker tests which identify significantly higher number of efficient investment opportunities. DEA tests with diversification were introduced by Briec et al. [10], Joro and Na [14], Lozano and Gutiérrez [16], Branda [3]. However, these models were limited to particular inputs and outputs. Recently, Lamb and Tee [15] employed positive parts of coherent risk measures as the inputs and return measures as the outputs and introduced a general class of diversification-consistent tests. Several extension were proposed by Branda [6] who introduced input and input-output oriented tests focusing on the strength of the tests. Moreover, he avoided cutting the negative part of risk measures and suggested general deviation measures (Rockafellar et al. [20]) to quantify riskiness as the inputs. The resulting models can be seen as generalizations of Markowitz mean-risk models [17, 18], where variance and semivariance were used to quantify risk and which both belong to the class of general deviation measures. Branda and Kopa [9] even showed equivalence between particular new DEA tests and stochastic dominance efficiency tests.
This paper is organized as follows. In Section 2, we will propose basic notation and axiomatic definitions of deviation and return measures. Standard DEA tests and diversification-consistent extensions are proposed in Section 3. In Section 4, we employ the DEA tests to access efficiency of 46 US industry representative portfolios.

2 Preliminaries and notation

In this section, we introduce basic notion and review the axioms of deviation and return measures, which will serve as the inputs and outputs in the following DEA tests. Let $X$ be a set of random returns of available investment opportunities corresponding to one single asset or to a portfolio consisting of many assets. We consider $n$ assets and denote $R_i$ the return of $i$-th asset which is a real random variable defined on the probability space $(\Omega, \mathcal{A}, P)$. We will use the set with no short sales enabling full diversification: $X = \{\sum_{i=1}^{n} R_i x_i : \sum_{i=1}^{n} x_i = 1, x_i \geq 0\}$. Other choices of the set are also possible, e.g. with limited number of assets, allowing short sales, borrowing or including proportional and fixed transaction costs, cf. Branda [4, 6].

Functionals $\mathcal{D} : \mathcal{L}_2(\Omega) \rightarrow [0, \infty]$ are called general deviation measures if they satisfy:

1. **Translation invariance**: $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all $X$ and constants $C$,
2. **Positive homogeneity**: $\mathcal{D}(X) = \mathcal{D}(\lambda X)$ for all $X$ and all $\lambda > 0$,
3. **Subadditivity**: $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all $X$ and $Y$,
4. **Nonnegativity**: $\mathcal{D}(X) \geq 0$ for all $X$, with $\mathcal{D}(X) > 0$ for nonconstant $X$.

Note that the axioms (D2) and (D3) imply convexity. The main examples are standard deviation and semideviations.

Functionals $\mathcal{E} : \mathcal{L}_p(\Omega) \rightarrow (-\infty, \infty]$ are called return measures if they satisfy:

1. **Translation equivariance**: $\mathcal{E}(X + C) = \mathcal{E}(X) + C$ for all $X$ and constants $C$,
2. **Positive homogeneity**: $\mathcal{E}(X) = \mathcal{E}(\lambda X)$ for all $X$ and all $\lambda > 0$,
3. **Superadditivity**: $\mathcal{E}(X + Y) \geq \mathcal{E}(X) + \mathcal{E}(Y)$ for all $X$ and $Y$,
4. **Monotonicity**: $\mathcal{E}(X) \geq \mathcal{E}(Y)$ when $X \geq Y$.

The space $\mathcal{L}_p(\Omega)$ is selected so as the measures are finite, usually $p = 1$. The axioms (E2) and (E3) imply concavity of the functional. It is obvious that expectation fulfills the axioms. Moreover, coherent risk measures multiplied by a negative constant can be used as return functionals, see Lamb and Tee [15].

3 DEA-risk tests

In this section, we will formulate several DEA tests, which will be employed in the numerical study to access efficiency of US market portfolios. We start with the standard model with variable return to scale. We can simply demonstrate its drawbacks and motivate an introduction of the DEA tests with diversification. The efficiency is expressed by the optimal value of the corresponding DEA tests – we say that an investment opportunity is efficient if the optimal value is equal to 1, otherwise we say that it is inefficient.

Traditional mean-risk (mean-deviation) efficiency can be defined as follows: an investment opportunity is efficient if there is no other investment opportunity with higher or equal expected return and lower or equal risk (deviation) with at least one inequality strict. This approach to efficiency is extended by the following DEA tests where $K$ deviation measures and $J$ return measures are employed at the same time.

3.1 Traditional input oriented tests

We are going to access efficiency of an investment opportunity $R_0$. We assume that the benchmark $R_0$ is not constant, which implies that the employed deviation measures are positive. The standard DEA test with variable return to scale introduced by Banker et al. [1] can be formulated in the dual form as
a linear program as follows:

\[
\theta_{VRS}(R_0) = \min_{\theta, x} \theta \\
\text{s.t. } \sum_{i=1}^{n} x_i E_j(R_i) \geq E_j(R_0), \ j = 1, \ldots, J, \\
\sum_{i=1}^{n} x_i D_k(R_i) \leq \theta \cdot D_k(R_0), \ k = 1, \ldots, K, \\
\sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \ 1 \geq \theta \geq 0.
\]

This dual formulation shows the main problem of the standard DEA tests in finance. It compares benchmark deviation with a linear combination of assets deviations, which has no financial or economical meaning. This drawback is removed by the diversification consistent (DC) tests proposed in the following parts.

3.2 Input oriented diversification-consistent tests

The input oriented test can be extended to take into account diversification. This can be simply done by interchanging the deviation and sum operators resulting into

\[
\theta_I(R_0) = \min_{\theta, x} \theta \\
\text{s.t. } E_j \left( \sum_{i=1}^{n} R_i x_i \right) \geq E_j(R_0), \ j = 1, \ldots, J, \\
D_k \left( \sum_{i=1}^{n} R_i x_i \right) \leq \theta \cdot D_k(R_0), \ k = 1, \ldots, K, \\
\sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \ 1 \geq \theta \geq 0.
\]

It can be shown that for convex deviation measures and concave return measures this test is stronger than the standard one, i.e. for any benchmark \( R_0 \) it holds \( \theta_{VRS}(R_0) \geq \theta_I(R_0) \) for arbitrary \( R_0 \in X \), see Branda [6]. We can obtain a mean-deviation efficiency test by restricting the number of considered inputs and outputs to one per class.

3.3 Input-output oriented diversification-consistent tests

We assume that \( E_j(R_0) \) is positive for at least one \( j \). An input-output oriented test where inefficiency is measured with respect to the inputs and outputs separately can be formulated as follows

\[
\theta_{I-O}(R_0) = \min_{\theta, \varphi, x} \theta \\
\text{s.t. } E_j \left( \sum_{i=1}^{n} R_i x_i \right) \geq \varphi \cdot E_j(R_0), \ j = 1, \ldots, J, \\
D_k \left( \sum_{i=1}^{n} R_i x_i \right) \leq \theta \cdot D_k(R_0), \ k = 1, \ldots, K, \\
\sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \ \varphi \geq 1, \ 1 \geq \theta \geq 0.
\]

The benchmark investment opportunity \( R_0 \) is efficient if and only if there is no other investment opportunity with lower or equal deviations and higher or equal returns with strict inequalities for at least one group. The optimal value can be seen as a ratio of minimal necessary improvement (=decrease) of inputs and minimal necessary improvement (=increase) of outputs to reach the efficiency.
The following reformulation was derived by Branda [7]. If we set \( 1/t = \varphi \) and substitute \( \tilde{x}_j = tx_i \), \( \tilde{\theta} = t\theta \), and \( \tilde{\varphi} = t\varphi \), the decision variables \( \tilde{\varphi} \) and \( t \) can be omitted resulting into an input oriented DEA test with nonincreasing return to scale (NIRS):

\[
\theta_{I-O}(R_0) = \min_{\tilde{\theta}, \tilde{x}_i} \tilde{\theta}
\]

s.t. \( E_j \left( \sum_{i=1}^{n} R_i \tilde{x}_i \right) \geq E_j(R_0), j = 1, \ldots, J, \)

\[
D_k \left( \sum_{i=1}^{n} R_i \tilde{x}_i \right) \leq \tilde{\theta} \cdot D_k(R_0), k = 1, \ldots, K,
\]

\[
\sum_{i=1}^{n} \tilde{x}_i \leq 1, \quad \tilde{x}_i \geq 0, \quad 1 \geq \tilde{\theta} \geq 0.
\]

Note that it is important for the reformulation that all inputs \( D_k \) and all outputs \( E_j \) are positively homogeneous. Since the general deviation measures are convex and the return measures are assumed to be concave, we obtained a convex programming problem. Similar class of input oriented tests was proposed by Lamb and Tee [15], where the assumption of NIRS was suggested for financial applications. Note that the input-output oriented test is stronger than the input oriented test, i.e. we obtain the relation between the optimal values \( \theta_1(R_0) \geq \theta_{I-O}(R_0) \) for arbitrary \( R_0 \in X \).

4 Numerical results

In this section, we employ the DEA tests proposed above to access efficiency of 46 US industry representative portfolios observed monthly from January 2002 to December 2011, see Branda and Kopa [9] for details. We consider discretely distributed returns \( r_{is}, s = 1, \ldots, S \) with equal probabilities \( 1/S \), in our case \( S = 120 \). For general continuous distributions we can obtain similar problems using sample approximation technique, cf. Branda [5]. CVaR deviation on level \( \alpha \) can be then formulated as

\[
D^S_{\alpha} \left( \sum_{i=1}^{n} R_i x_i \right) = \min_{\xi \in X} \frac{1}{S} \sum_{s=1}^{S} \max \left\{ \frac{1}{S} \left( \sum_{i=1}^{n} x_i r_{is} - \xi_s \right), \frac{\alpha}{1 - \alpha} \left( \xi - \sum_{i=1}^{n} x_i r_{is} \right) \right\}
\]

We consider \( D^S_{\alpha_k} \) for various levels \( \alpha_k \in (0, 1), k = 1, \ldots, K \), in particular \( \alpha_k \in \{0.75, 0.9, 0.95, 0.99\} \) and \( K = 4 \), as the inputs and the expectation as an output, i.e. \( J = 1 \) and \( E_1(X) = E(X) \). This leads to the following linear programming problem for the input-output oriented test:

\[
\theta_{I-O}(R_0) = \min_{\theta, x_i, u_{sk}, \xi_k} \theta
\]

s.t. \( \sum_{i=1}^{n} E[R_i] x_i \geq E[R_0], \)

\[
\frac{1}{S} \sum_{s=1}^{S} u_{sk} \leq \theta \cdot D^S_{\alpha_k}(R_0), k = 1, \ldots, K,
\]

\[
u_{sk} \geq \left( \sum_{i=1}^{n} x_i r_{is} - \xi_k \right), s = 1, \ldots, S, k = 1, \ldots, K,
\]

\[
u_{sk} \geq \frac{\alpha_k}{1 - \alpha_k} \left( \xi_k - \sum_{i=1}^{n} x_i r_{is} \right), s = 1, \ldots, S, k = 1, \ldots, K,
\]

\[
\sum_{i=1}^{n} x_i \leq 1, \quad x_i \geq 0, \quad i = 1, \ldots, n.
\]

Similar reformulations were obtained by Branda [6, 7] for the input oriented and input-output oriented DEA tests with general probabilities. However, no numerical comparison of these approaches was reported in the previous papers.

We solved the DEA problems by the CPLEX 12.1 solver using the modeling system GAMS 23.2. Selected efficiency scores can be found in Table 1. We selected only the seven portfolios which are
efficient according to the VRS DEA test which is the weakest. The stronger DC tests identified only the portfolio Coal as efficient. Most of the VRS efficient portfolios are highly ranked also by the DC tests. The only exception is Drugs, which is ranked by the input-output oriented DC test very badly. We also computed correlations between the optimal values. Perhaps surprisingly we observed higher correlation between the efficiency scores of the VRS and input oriented test (0.97) than between the DC tests (0.70). The simplest explanation is the common orientation on inputs. Ranking of all representative portfolios can be found in Table 2.

<table>
<thead>
<tr>
<th></th>
<th>Food</th>
<th>Smoke</th>
<th>Hshld</th>
<th>Drugs</th>
<th>Mines</th>
<th>Coal</th>
<th>Meals</th>
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<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
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<tr>
<td>DC Inp</td>
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Table 1 Efficient industry representative portfolios and scores

5 Conclusions

In this paper, we have compared several approaches to efficiency of investment opportunities based on DEA. The traditional DEA test with variable return to scale has been shown much weaker that the extensions with diversification. These diversification consistent tests identified only one portfolio as efficient. However, the resulting rankings are significantly different. The diversification-consistent tests can be also seen as superefficiency models for the standard VRS DEA tests saying how much the benchmark inputs and outputs need to be improved to reach the efficiency frontier. Future research will be devoted to multiperiod–dynamic extensions.

<table>
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<tr>
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<th>Agric</th>
<th>Food</th>
<th>Soda</th>
<th>Beer</th>
<th>Smoke</th>
<th>Toys</th>
<th>Fun</th>
<th>Hshld</th>
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<tr>
<td>MedEq</td>
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Table 2 Ranking of the industry representative portfolios

Acknowledgements

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References


