Chance constrained problems: reformulation using penalty functions and sample approximation technique

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   - Program with a random factor
     - Chance constrained problem (CCP) ⇔ Problem with penalty objective (PPO) ⇔ Integrated chance constrained problem (ICC)

2. Sample approximation (S.A.)

   - S.A. CCP → S.A. PPO → S.A. ICC

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Program with a random factor $\xi$

$$\min \{ f(x) : x \in X, g_i(x, \xi) \leq 0, \ i = 1, \ldots, k \},$$

where $g_i, i = 0, \ldots, k,$ are real functions on $\mathbb{R}^n \times \mathbb{R}^{n'}, X \subseteq \mathbb{R}^n$ and $\xi \in \mathbb{R}^{n'}$ is a realization of a $n'$-dimensional random vector defined on the probability space $(\Omega, \mathcal{F}, P)$.

If $P$ is known, we can use chance constraints to deal with the random constraints...
Stochastic programming formulations

Chance constrained problem (CCP)

**Chance constrained problem**

\[
\psi_\epsilon = \min_{x \in X} f(x),
\]

s.t.

\[
P(g_{11}(x, \xi) \leq 0, \ldots, g_{1k_1}(x, \xi) \leq 0) \geq 1 - \epsilon_1, \\
\vdots \\
P(g_{m1}(x, \xi) \leq 0, \ldots, g_{mk_m}(x, \xi) \leq 0) \geq 1 - \epsilon_m,
\]

with optimal solution \(x_\epsilon\), where we denoted \(\epsilon = (\epsilon_1, \ldots, \epsilon_m)\) with levels \(\epsilon_j \in (0, 1)\). The formulation covers the joint \((k_1 > 1\text{ and } m = 1)\) as well as the individual \((k_j = 1\text{ and } m > 1)\) chance constrained problems as special cases.
In general,

- the feasible region is **not convex** even if the functions are convex,
- it is even not easy to check **feasibility** because it leads to computations of multivariate integrals.

Hence, we will try to reformulate the chance constrained problem using penalty functions.
Penalty functions $\vartheta_j : \mathbb{R}^m \rightarrow \mathbb{R}_+$, $j = 1, \ldots, m$, are continuous nondecreasing, equal to 0 on $\mathbb{R}^m_-$ and positive otherwise, e.g.

$$\vartheta^1_p(u) = \sum_{i=1}^{k} ([u_i]^+)^p, \quad p \in \mathbb{N}$$

$$\vartheta^2(u) = \max_{1 \leq i \leq k} [u_i]^+,$$

$$= \min \left\{ t \geq 0 : u_i - t \leq 0, \quad i = 1, \ldots, k \right\}$$

where $u \in \mathbb{R}^m$. Let $p_j$ denote the penalized constraints

$$p_j(x, \xi) = \vartheta_j(g_{j1}(x, \xi), \ldots, g_{jk_j}(x, \xi)), \forall j.$$
Penalty function problems

Problem with **penalties in the objective function**

\[
\varphi_N = \min_{x \in X} \left\{ f(x) + N \cdot \sum_{j=1}^{m} \mathbb{E}[p_j(x, \xi)] \right\}
\]

with an optimal solution \(x_N\). In Ermoliev et al. (2000) for \(\vartheta^{1,1}\) and \(m = 1\).

Problem with **generalized integrated chance constraints**

\[
\varphi_{L}^{\text{ICC}} = \min_{x \in X} \left\{ f(x) : \text{ s.t. } \mathbb{E}[p_j(x, \xi)] \leq L_j, j = 1, \ldots, m \right\}
\]

for some prescribed bounds \(L_j \geq 0\), \(L = (L_1, \ldots, L_m)^T\), with an optimal solution \(x_{L}^{\text{ICC}}\) (originally defined using \(u^2\), cf. Klein Haneveld (1986)).
Penalty function problems

Problem with **penalties in the objective function**

\[ \varphi_N = \min_{x \in X} \left\{ f(x) + N \cdot \sum_{j=1}^{m} \mathbb{E}[p_j(x, \xi)] \right\} \]

with an optimal solution \( x_N \). In Ermoliev et al. (2000) for \( \vartheta^{1,1} \) and \( m = 1 \).

Problem with **generalized integrated chance constraints**

\[ \varphi_{L}^{ICC} = \min_{x \in X} \left\{ f(x) : \text{s.t. } \mathbb{E}[p_j(x, \xi)] \leq L_j, j = 1, \ldots, m \right\} \]

for some prescribed bounds \( L_j \geq 0, \ L = (L_1, \ldots, L_m)' \), with an optimal solution \( x_{L}^{ICC} \) (originally defined using \( u^2 \), cf. Klein Haneveld (1986)).
Stochastic programming formulations

History and applications of the penalty approach in SP

- Prékopa (1973): **CPP and penalization**
- Ermoliev et al (2000): Managing exposure to **catastrophic risks** (asymptotic equivalence with particular penalty)
- Žampachová (2009): **Beam design** (reliability problem with partial differential equations - nonlinear - significant reduction of computational time)
- M.B. (2009, 2012A): **Value at Risk optimization** with transaction costs and integer allocations (general penalty functions and several CC)
- M.B (2011): **Blending problem** (asymptotic equivalence with generalized integrated chance constraints)
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Relations between formulations

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   - S.A. CCP
   - S.A. PPO
   - S.A. ICC

3. Solution validation
   - Reliability check
   - Reliability check
   - Reliability check
M.B. (2012A): Under the following assumptions, the **asymptotic equivalence of the CCP and the PPO problems** can be shown:

- **Continuity** of the constraints and the probabilistic functions.
- **Compactness** of the fixed set of feasible solutions.
- Existence of **integrable majorants**.
- Existence of a **permanently feasible solution**.

THEN for any prescribed $\epsilon \in (0, 1)^m$ there always exists $N$ large enough so that minimization of the penalty objective generates optimal solutions $x_N$ which also satisfy the chance constraints with the given $\epsilon$. 
Denote $\eta = \kappa/(2(1 + \kappa))$, and for arbitrary $N > 0$ and $\epsilon \in (0, 1)^m$ put

$$
\epsilon_j(x) = P(p_j(x, \xi) > 0), \ j = 1, \ldots, m,
$$

$$
\alpha_N(x) = N \cdot \sum_{j=1}^{m} \mathbb{E}[p_j(x, \xi)], \ \beta_\epsilon(x) = \epsilon_{\max}^{-\eta} \sum_{j=1}^{m} \mathbb{E}[p_j(x, \xi)],
$$

where $\epsilon_{\max} = \max_j \epsilon_j$ and $[1/N^{1/\eta}] = (1/N^{1/\eta}, \ldots, 1/N^{1/\eta})$. THEN bounds on the optimal values can be constructed:

$$
\varphi_{[1/N^{1/\eta}]}(x_N) - \beta_\epsilon(x_N)(x_\epsilon(x_N)) \leq \psi_\epsilon(x_N) \leq \varphi_N - \alpha_N(x_N),
$$

$$
\psi_\epsilon(x_N) + \alpha_N(x_N) \leq \varphi_N \leq \psi_{[1/N^{1/\eta}]} + \beta_{[1/N^{1/\eta}]}(x_{[1/N^{1/\eta}]}).
$$

with

$$
\lim_{N \to +\infty} \alpha_N(x_N) = \lim_{N \to +\infty} \epsilon_j(x_N) = \lim_{\epsilon_{\max} \to 0^+} \beta_\epsilon(x_\epsilon) = 0
$$

for any sequences of optimal solutions $x_N$ and $x_\epsilon$. 

M. Branda (Charles University)
Relations between formulations

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3. Solution validation
   - Reliability check
M.B. (2011): Under the following assumptions, the asymptotic equivalence of the ICC and the PPO problems can be shown:

- **Continuity** of the constraints.
- **Compactness** of the fixed set of feasible solutions.
- Existence of **integrable majorants**.
- Existence of a **permanently feasible solution**.

THEN for any prescribed $L_j \geq 0$ there always exists $N$ large enough so that minimization of the penalty function problem generates the optimal solutions $x_N$ which also satisfy the integrated chance constraints with given $L = (L_1, \ldots, L_m)'$. 
For arbitrary $\gamma \in (0, 1)$, $N > 0$ and $L_j \geq 0$ put

$$L_j(x) = \mathbb{E}[p_j(x, \xi)], \quad j = 1, \ldots, m,$$

$$\alpha_N(x) = N \cdot \sum_{j=1}^{m} \mathbb{E}[p_j(x, \xi)], \quad \beta_L(x) = \left( \sum_{j=1}^{m} L_j \right)^{\gamma-1} \sum_{j=1}^{m} \mathbb{E}[p_j(x, \xi)],$$

and let $\left[ N^{1/(\gamma-1)}/m \right] = \left( N^{1/(\gamma-1)}/m, \ldots, N^{1/(\gamma-1)}/m \right)'$. THEN bounds on the optimal values can be constructed:

$$\varphi \left( \sum_{j=1}^{m} L_j(x_N) \right)^{\gamma-1} - \beta_L(x_N) \left( x_L(x_N) \right) \leq \varphi^{ICC}_L(x_N) \leq \varphi_N - \alpha_N(x_N),$$

$$\varphi^{ICC}_L(x_N) + \alpha_N(x_N) \leq \varphi_N \leq \varphi^{ICC}_{\left[ N^{1/(\gamma-1)}/m \right]} + \beta_{\left[ N^{1/(\gamma-1)}/m \right]} \left( x_{\left[ N^{1/(\gamma-1)}/m \right]}^{ICC} \right),$$

with

$$\lim_{N \to +\infty} \alpha_N(x_N) = \lim_{N \to +\infty} L_j(x_N) = \lim_{L_{\text{max}} \to 0^+} \beta_L(x_L^{ICC}) = 0$$

for any sequences of the optimal solutions $x_N$ and $x_{L}^{ICC}$ where $L_{\text{max}}$ denotes the maximal component of the vector $L$.\"
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### Sample approximations using Monte-Carlo techniques

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Let $\xi^1, \ldots, \xi^S$ be an independent Monte Carlo sample of the random vector $\xi$.

By generalization of the results proposed by S. Ahmed, J. Luedtke, A. Shapiro, et al. (2008, 2009) ...
Sample approximations using Monte-Carlo techniques

Sample approximated chance constrained problem

...can be reformulated as a **large mixed-integer nonlinear program**:

\[
\begin{align*}
\min_{(x,u) \in X \times \{0,1\}^S} f(x) \\
\text{s.t.} & \\
& g_{1i}(x, \xi^s) - M(1 - u_{1s}) \leq 0, \ i = 1, \ldots, k_1, \ s = 1, \ldots, S \\
& \quad \vdots \\
& g_{mi}(x, \xi^s) - M(1 - u_{ms}) \leq 0, \ i = 1, \ldots, k_m, \ s = 1, \ldots, S, \\
& \frac{1}{S} \sum_{s=1}^{S} u_{1s} \geq 1 - \gamma_1, \\
& \quad \vdots \\
& \frac{1}{S} \sum_{s=1}^{S} u_{ms} \geq 1 - \gamma_m, \\
& u_{1s}, \ldots, u_{ms} \in \{0, 1\}, \ s = 1, \ldots, S,
\end{align*}
\]

where $M$ is a large constant and $\gamma_j \in (0, 1)$. 
M.B. (2012A): Let $\gamma_j > \varepsilon_j$ for all $j$, where $\gamma_j$ are levels used in S.A. problems.

Sample size $S$ necessary to obtain that a feasible solution of the original CC problem is also feasible for the sample approximation with a probability at least $1 - \delta$, $\delta \in (0, 1)$ small:

$$S \geq \frac{2}{\min_{j \in \{1, \ldots, m\}}(\gamma_j - \varepsilon_j)^2 / \varepsilon_j} \ln \frac{m}{\delta}.$$  


(The estimate is based on Chernoff and Bonferroni inequalities.)
M.B. (2012A): Let $\gamma_j < \varepsilon_j$ for all $j$ and $|X \setminus X_\varepsilon|$ denote the number of points from $X$ which are not feasible for the original CC problem.

Sample size $S$ necessary for that the feasible solutions of the sample approximated problems are feasible for the original CC problem with a high probability $1 - \delta$:

$$S \geq \frac{1}{2 \min_{j \in \{1, \ldots, m\}} (\gamma_j - \varepsilon_j)^2} \ln \frac{m |X \setminus X_\varepsilon|}{\delta}.$$  

If we set $m = 1$, we get the same inequality as J. Luedtke, et al (2008).

(The estimate is based on Hoeffding and Bonferroni inequalities.) Extended for the **bounded infinite** and **mixed-integer** set of feasible solutions, see M.B. (2012A, 2012B)...
Mixed-integer CCP

\[
\min_{(x,y) \in \mathbb{Z}} f(x, y), \\
\text{s.t.} \\
P\left( g_{11}(x, y, \xi) \leq 0, \ldots, g_{1k_1}(x, y, \xi) \leq 0 \right) \geq 1 - \varepsilon_1, \\
\vdots \\
P\left( g_{m1}(x, y, \xi) \leq 0, \ldots, g_{mk_m}(x, y, \xi) \leq 0 \right) \geq 1 - \varepsilon_m, 
\]

where \( \varepsilon_j \in (0, 1) \), \( X \subseteq \mathbb{R}^n \), \( Y \subseteq \mathbb{Z}^{n'} \) and
\[
Z = \{ (x, y) \in X \times Y : h_1(x, y) \leq 0, \ldots, h_k(x, y) \leq 0 \}, \\
g_{ji}(x, y, \xi) : \mathbb{R}^n \times \mathbb{Z}^{n'} \times \mathbb{R}^{n''} \to \mathbb{R}, i = 0, \ldots, k_j, j = 1, \ldots, m \text{ measurable in } \xi \text{ for all } x \in X \text{ and } y \in Y, \\
f(x, y) : \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}.
\]
M.B. (2012B): Let

1. $\gamma_j < \varepsilon_j$, i.e. that the levels of the sample approximated problem are more restrictive,

2. $Y \subseteq \mathbb{Z}^{n'}$ be finite,

3. $X(y) = \{x \in X : (x, y) \in Z\}$ be uniformly bounded for all $y \in Y$, i.e. $D = \sup_{y \in Y} \sup \{\|x - x'\|_{\infty} : x, x' \in X(y)\}$ be a finite diameter,

4. functions $G_j(x, y, \omega) = \max\{g_{j1}(x, y, \omega), \ldots, g_{jk_j}(x, y, \omega)\}$ be Lipschitz continuous in the real variable $x$, i.e. for arbitrary $y \in Y$ and $\xi \in \Xi$

$$|G_j(x, y, \omega) - G_j(x', y, \omega)| \leq L_j \|x - x'\|_{\infty}, \forall x, x' \in X(y),$$

for some $L_j > 0.$
Mixed-integer CCP

M.B. (2012B): It is possible to estimate the sample size $S$ such that the feasible solutions of the relaxed sample-approximated problems are feasible for the original problem with a high probability $1 - \delta$:

$$S \geq \frac{1}{2 \min_j (\varepsilon_j - \gamma_j - \lambda_j)^2} \left( \ln \frac{m|Y|^2}{\delta} + \ln \left\lfloor \frac{1}{\lambda_{min}} \right\rfloor + n \ln \left\lfloor \frac{2L_{max}D}{\tau} \right\rfloor \right),$$

where $L_{max} = \max_j L_j$ and $\lambda_{min} = \min_j \lambda_j$, $\lambda_j \in (0, \varepsilon_j - \gamma_j)$, $\tau > 0$ small.

In M.B. (2012B) applied to stochastic vehicle routing problem.
Sample approximations using Monte-Carlo techniques

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| 2. Sample approximation (S.A.) |
| S.A. CCP | S.A. PPO | S.A. ICC |
| \(\downarrow\) | \(\downarrow\) | \(\downarrow\) |

| 3. Solution validation |
| Reliability check | Reliability check | Reliability check |
| \(\downarrow\) | \(\downarrow\) | \(\downarrow\) |
The set of feasible solutions of the original ICC problem

\[ X_L = \left\{ x \in X : p_j(x) := \mathbb{E}[p_j(x, \xi)] \leq L_j, \ j = 1, \ldots, m \right\}, \]

the (relaxed) set of feasible solutions of the sample approximated problem, \( \tau_j \in \mathbb{R}, \ \tau = (\tau_1, \ldots, \tau_m) \),

\[ X_{L+\tau}^S = \left\{ x \in X : p_j^S(x) := \frac{1}{S} \sum_{s=1}^{S} p_j(x, \xi^s) \leq L_j + \tau_j, \ j = 1, \ldots, m \right\}. \]
M.B. (2011): Let the moment generating function of \( p_j(x, \xi) - p_j(x) \) is finite and \( \tau_j > 0 \).

Estimated sample size \( S \) such that the feasible solutions of the original problem are feasible for the relaxed sample approximated problem with a high probability \( 1 - \delta \):

\[
S \geq \frac{1}{\min_{j \in \{1, \ldots, m\}, x \in X} \frac{\tau_j^2}{2\sigma_{jx}^2}} \ln \frac{m|X|}{\delta},
\]

where \( \sigma_{jx}^2 = \text{Var}[p_j(x, \xi) - p_j(x)] < \infty \). If we set \( m = 1 \), we get similar inequality as W. Wang, S. Ahmed (2008).

(Based on Large Deviation Theory and Bonferroni inequality.) Extended for the \textit{bounded infinite set} of feasible solutions, see M.B. (2011).
Sample approximations using Monte-Carlo techniques

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