Data Envelopment Analysis in Finance

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1. Efficiency of investment opportunities
2. Data Envelopment Analysis
3. Diversification-consistent DEA based on general deviation measures
   - General deviation measures
   - Diversification-consistent DEA models
   - Financial indices efficiency – empirical study
4. On relations between DEA and stochastic dominance efficiency
   - Second Order Stochastic Dominance
   - Data Envelopment Analysis
   - Numerical comparison
5. References
- We do not access efficiency of financial institutions (banks, insurance comp.).
- We access efficiency of investment opportunities\(^1\) on financial markets.

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\(^1\) Assets, portfolios, mutual funds, financial indices, ...
Motivation

Together with Miloš Kopa (in 2010):

Is there a relation between stochastic dominance efficiency and DEA efficiency?

Could we benefit from the relation?

- **DEA** – traditional strong wide area (many applications and theory, Handbooks, papers in highly impacted journals, e.g. Omega, EJOR, JOTA, JORS, EE, JoBF)
- **Stochastic dominance** – quickly growing area in finance and optimization
- Branda, Kopa (2012): an empirical study (a bit “naive”, but necessary step for us:)
- Branda, Kopa (2014): equivalences (a “bridge”)

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DEA in Finance  
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Various approaches how to find an “optimal” portfolio or how to test efficiency of an investment opportunity:

- von Neumann and Morgenstern (1944): Utility, expected utility
- Markowitz (1952): Mean-variance, mean-risk, mean-deviation
- Murthi et al (1997): Data Envelopment Analysis
This presentation contains

- **DEA efficiency** in finance – Murthi et al. (1997), Briec et al. (2004), Lamb and Tee (2012)

- Extension of **mean-risk efficiency** based on **multiobjective optimization** principles – Markowitz (1952)

- **Risk shaping** — several risk measures (CVaRs) included into one model – Rockafellar and Uryasev (2002)

- Relations to **stochastic dominance** efficiency – Branda and Kopa (2014)
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Charnes, Cooper and Rhodes (1978): a way how to state efficiency of a decision making unit over all other decision making units with the same structure of inputs and outputs.

Let $Z_{1i}, \ldots, Z_{Ki}$ denote the inputs and $Y_{1i}, \ldots, Y_{Ji}$ denote the outputs of the unit $i$ from $n$ considered units. **DEA efficiency** of the unit $0 \in \{1, \ldots, n\}$ is then evaluated using the optimal value of the following program where the weighted inputs are compared with the weighted outputs.

All data are assumed to be (semi-)positive. Charnes et al (1978): fractional programming formulation (Constant Returns to Scale – CRS or CCR)
Banker, Charnes and Cooper (1984): DEA model with Variable Returns to Scale (VRS) or BCC:

$$\max_{y_{j0}, w_{k0}} \frac{\sum_{j=1}^{J} y_{j0} Y_{j0} - y_0}{\sum_{k=1}^{K} w_{k0} Z_{k0}}$$

s.t.

$$\frac{\sum_{j=1}^{J} y_{j0} Y_{ji} - y_0}{\sum_{k=1}^{K} w_{k0} Z_{ki}} \leq 1, \ i = 1, \ldots, n,$$

$$w_{k0} \geq 0, \ k = 1, \ldots, K,$$

$$y_{j0} \geq 0, \ j = 1, \ldots, J,$$

$$y_0 \in \mathbb{R}.$$
Data Envelopment Analysis

DEA
Variable Returns to Scale (VRS)

Dual formulation of VRS DEA:

\[
\begin{align*}
\min_{\theta, x_i} & \quad \theta \\
\text{s.t.} & \\
\sum_{i=1}^{n} x_i Y_{ji} & \geq Y_{j0}, \ j = 1, \ldots, J, \\
\sum_{i=1}^{n} x_i Z_{ki} & \leq \theta \cdot Z_{k0}, \ k = 1, \ldots, K, \\
\sum_{i=1}^{n} x_i & = 1, \ x_i \geq 0, \ i = 1, \ldots, n.
\end{align*}
\]
production theory (production possibility set),
returns to scale (CRS, VRS, NIRS, ...),
radial/slacks-based/directional distance models,
fractional/primal/dual formulations,
multiobjective opt. – strong/weak Pareto efficiency,
stochastic data – reliability, chance constraints,
dynamic (network) DEA,
super-efficiency, cross-efficiency, ...
the most efficient unit
...
DEA efficiency corresponds to multiobjective (Pareto) efficiency where
- all inputs are minimized
- and/or all outputs are maximized
- under some conditions.
Efficiency of mutual funds or financial indexes:

- Basso and Funari (2001, 2003): standard deviation and semideviations, beta coefficient, costs as the inputs, expected return or expected excess return, ethical measure and stochastic dominance criterion as the outputs.
- Chen and Lin (2006): Value at Risk (VaR) and Conditional Value at Risk (CVaR).
- Branda and Kopa (2012): VaR, CVaR, sd, lsd, Drawdow measures (DaR, CDaR) as the inputs, gross return as the output; comparison with second-order stochastic dominance.

See Table 1 in Lozano and Gutiérrez (2008B)
Lamb and Tee (2012) – pure return-risk tests$^2$:

- **Inputs**: positive parts of coherent risk measures
- **Outputs**: return measures (= minus coherent risk measures, e.g. expected return)

$^2$no transactions costs etc.
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General deviation measures

Rockafellar, Uryasev and Zabarankin (2006A, 2006B): GDM are introduced as an extension of *standard deviation* but they need not to be symmetric with respect to *upside* $X - \mathbb{E}[X]$ and *downside* $\mathbb{E}[X] - X$ of a random variable $X$.

Any functional $\mathcal{D} : L_2(\Omega) \rightarrow [0, \infty]$ is called a general deviation measure if it satisfies

- (D1) $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all $X$ and constants $C$,
- (D2) $\mathcal{D}(0) = 0$, and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all $X$ and all $\lambda > 0$,
- (D3) $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all $X$ and $Y$,
- (D4) $\mathcal{D}(X) \geq 0$ for all $X$, with $\mathcal{D}(X) > 0$ for nonconstant $X$.

(D2) & (D3) $\Rightarrow$ convexity
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Deviation measures

- **Standard deviation**
  \[ D(X) = \sigma(X) = \sqrt{\mathbb{E} \|X - \mathbb{E}[X]\|_2} \]

- **Mean absolute deviation**
  \[ D(X) = \mathbb{E}[|X - \mathbb{E}[X]|] \]

- **Mean absolute lower and upper semideviation**
  \[ D_{-}(X) = \mathbb{E}[|X - \mathbb{E}[X]|_{-}], \quad D_{+}(X) = \mathbb{E}[|X - \mathbb{E}[X]|_{+}] \]

- **Worst-case deviation**
  \[ D(X) = \sup_{\omega \in \Omega} |X(\omega) - \mathbb{E}[X]|. \]

Mean absolute deviation from \((1 - \alpha)\)-th quantile

CVaR deviation

For any \(\alpha \in (0, 1)\) a finite, continuous, lower range dominated deviation measure

\[
D_\alpha(X) = CVaR_\alpha(X - \mathbb{E}[X]).
\]  

(1)

The deviation is also called \textbf{weighted mean absolute deviation from the \((1 - \alpha)\)-th quantile}, see Ogryczak, Ruszczynski (2002), because it can be expressed as

\[
D_\alpha(X) = \min_{\xi \in \mathbb{R}} \frac{1}{1 - \alpha} \mathbb{E}\left[ \max\{(1 - \alpha)(X - \xi), \alpha(\xi - X)\} \right]
\]  

(2)

with the minimum attained at any \((1 - \alpha)\)-th quantile. In relation with CVaR minimization formula, see Pflug (2000), Rockafellar and Uryasev (2000, 2002).
According to Proposition 4 in Rockafellar et al (2006 A):

- if $\mathcal{D} = \lambda \mathcal{D}_0$ for $\lambda > 0$ and a deviation measure $\mathcal{D}_0$, then $\mathcal{D}$ is a deviation measure.

- if $\mathcal{D}_1, \ldots, \mathcal{D}_K$ are deviation measures, then
  - $\mathcal{D} = \max\{\mathcal{D}_1, \ldots, \mathcal{D}_K\}$ is also deviation measure.
  - $\mathcal{D} = \lambda_1 \mathcal{D}_1 + \cdots + \lambda_K \mathcal{D}_K$ is also deviation measure, if $\lambda_k > 0$ and $\sum_{k=1}^K \lambda_k = 1$.

Coherent risk and return measures

CRM: \( \mathcal{R} : \mathcal{L}_2(\Omega) \rightarrow (-\infty, \infty] \) that satisfies

(R1) \( \mathcal{R}(X + C) = \mathcal{R}(X) - C \) for all \( X \) and constants \( C \),

(R2) \( \mathcal{R}(0) = 0 \), and \( \mathcal{R}(\lambda X) = \lambda \mathcal{R}(X) \) for all \( X \) and all \( \lambda > 0 \),

(R3) \( \mathcal{R}(X + Y) \leq \mathcal{R}(X) + \mathcal{R}(Y) \) for all \( X \) and \( Y \),

(R4) \( \mathcal{R}(X) \leq \mathcal{R}(Y) \) when \( X \geq Y \).

Strictly expectation bounded risk measures satisfy (R1), (R2), (R3), and

(R5) \( \mathcal{R}(X) > \mathbb{E}[-X] \) for all nonconstant \( X \), whereas \( \mathcal{R}(X) = \mathbb{E}[-X] \) for constant \( X \).

Coherent risk and return measures

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$\mathcal{R}(X) = \mathbb{E}[-X]$ for constant $X$.

Other classes of risk measures and functionals: Follmer and Schied (2002),
A return measure is defined as a functional $\mathcal{E}(X) = -\mathcal{R}(X)$ for a coherent risk measure $\mathcal{R}$. It is obvious that the expectation belongs to this class. The right part of a return distribution described better by return measures derived from CVaR, cf. Lamb and Tee (2012).
We say that general deviation measure $\mathcal{D}$ is

- **(LSC) lower semicontinuous** (lsc) if all the subsets of $L_2(\Omega)$ having the form $\{X : \mathcal{D}(X) \leq c\}$ for $c \in \mathbb{R}$ (level sets) are closed;

- **(D5) lower range dominated** if $\mathcal{D}(X) \leq EX - \inf_{\omega \in \Omega} X(\omega)$ for all $X$. 
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Strictly expectation bounded risk measures

Theorem 2 in Rockafellar et al (2006 A):

**Theorem**

*Deviation measures correspond one-to-one with strictly expectation bounded risk measures under the relations*

- $\mathcal{D}(X) = \mathcal{R}(X - \mathbb{E}[X])$
- $\mathcal{R}(X) = \mathbb{E}[-X] + \mathcal{D}(X)$

*In this correspondence, $\mathcal{R}$ is coherent if and only if $\mathcal{D}$ is lower range dominated.*
Traditional DEA model
Input oriented (VRS)

We assume that $X_0$ is not constant, i.e. $D_k(X_0) > 0$, for all $k = 1, \ldots, K$. Input oriented VRS model can be formulated in the dual form

$$\theta_0^l(X_0) = \min \theta$$

s.t.

$$\sum_{i=1}^{n} x_i E_j(R_i) \geq E_j(X_0), \quad j = 1, \ldots, J,$$

$$\sum_{i=1}^{n} x_i D_k(R_i) \leq \theta \cdot D_k(X_0), \quad k = 1, \ldots, K,$$

$$\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \quad i = 1, \ldots, n.$$
The model does not take into account portfolio diversification: For any general deviation measure $D_k$ it holds

$$\sum_{i=1}^{n} x_i D_k(R_i) \geq D_k\left(\sum_{i=1}^{n} x_i R_i\right)$$

for nonnegative weights with $\sum_{i=1}^{n} x_i = 1$.

Linear transformation of inputs is only an upper bound for the real portfolio deviation.
The model does not take into account portfolio diversification: For any general deviation measure $\mathcal{D}_k$ it holds

$$\sum_{i=1}^{n} x_i \mathcal{D}_k(R_i) \geq \mathcal{D}_k\left(\sum_{i=1}^{n} x_i R_i\right)$$

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Linear transformation of inputs is only an upper bound for the real portfolio deviation.
Traditional DEA and diversification frontier

![Graph showing Expected return vs CVaR0.95 with Mean-CVaR eff. frontier, DEA eff. frontier, and portfolios marked.]

- Mean-CVaR eff. frontier
- DEA eff. frontier
- Portfolios
Efficiency of mutual funds or financial indexes using DEA related models with diversification:

- Kopa (2011): Comparison of several approaches (VaR, CVaR, ...).
- Branda (2011): models consistent with TSD.
- Lamb and Tee (2012): input-oriented diversification consistent model with several inputs and outputs (positive parts of risk measures used as the inputs).
DEA tests with diversification

Efficiency of mutual funds or industry representative portfolios:

Set of investment opportunities

We consider \( n \) assets and denote \( R_i \in \mathcal{L}_2(\Omega) \) the rate of return of \( i \)-th asset and the sets of investment opportunities:

1. **pairwise efficiency** (investment into one single asset):
   \[ \mathcal{X}^P = \{ R_i, i = 1, \ldots, n \}, \]

2. **full diversification** (diversification across all assets):
   \[ \mathcal{X}^{FD} = \left\{ \sum_{i=1}^{n} R_i x_i : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \right\}, \]

3. **limited diversification** (diversification across limited number of assets \#):
   \[ \mathcal{X}^{LD} = \left\{ \sum_{i=1}^{n} R_i x_i : \sum_{i=1}^{n} x_i = 1, x_i \geq 0, x_i \leq y_i, y_i \in \{0, 1\}, \sum_{i=1}^{n} y_i \leq \# \right\}. \]
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   \]

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   \[
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   \[
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   \]
Partial ordering of vectors

Definition

Let $v, z \in \mathbb{R}^n$. We say that $z$ strictly dominates (weakly) $v$, denoted

- $v \prec_{st} z$ if $\forall i : v_i < z_i$,
- $v \prec_w z$ if $\forall i : v_i \leq z_i$ and $\exists \tilde{i} : v_{\tilde{i}} < z_{\tilde{i}}$.

Definition

Let $v, z \in \mathbb{R}^n$. We say that $z$ partially strictly (partially weakly) dominates $v$ with respect to an index set $S \subseteq \{1, \ldots, n\}$, denoted

- $v \prec_{pst(S)} z$ if $v_i < z_i$ for all $i \in S$ and $v_i \leq z_i$ for all $i \in \{1, \ldots, n\} \setminus S$,
- $v \prec_{pw(S)} z$ if $v_i \leq z_i$ for all $i \in \{1, \ldots, n\}$ and there exists at least one $\tilde{i} \in S$ for which $v_{\tilde{i}} < z_{\tilde{i}}$. 

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We assume that \( X_0 \in \mathcal{X} \) is not constant, i.e. \( D_k(X_0) > 0 \), for all \( k = 1, \ldots, K \).

**Definition**

We say that \( X_0 \in \mathcal{X} \) is **DEA efficient** with respect to the set \( \mathcal{X} \) if the optimal value of the DEA program is equal to 1. Otherwise, \( X_0 \) is inefficient and the optimal value measures the inefficiency.

For \( m \in \{0, 1, 2, 3\} \) we denote the sets of efficient opportunities

\[
\Psi^I_m = \{ X \in \mathcal{X} : \theta^I_m(X) = 1 \},
\]

where \( \theta^I_m(X_0) \) is the optimal value for benchmark \( X_0 \).
Diversification consistent DEA - model 1

For a benchmark $X_0 \in \mathcal{X}$ diversification consistent DEA model

$$\theta_1^l(X_0) = \min \theta$$

s.t.

$$\mathcal{E}_j(X) \geq \mathcal{E}_j(X_0), \quad j = 1, \ldots, J,$$

$$\mathcal{D}_k(X) \leq \theta \cdot \mathcal{D}_k(X_0), \quad k = 1, \ldots, K,$$

$$X \in \mathcal{X}.$$
Set of efficient opportunities

Proposition

\( X_0 \in \Psi_1 \)

- if for all \( X \in \mathcal{X} \) for which \( \mathcal{E}_j(X) \geq \mathcal{E}_j(X_0) \) for all \( j \) holds that
  \( \mathcal{D}_k(X) \geq \mathcal{D}_k(X_0) \) for at least one \( k \), i.e. there is no \( X \in \mathcal{X} \) for which
  \( \mathcal{E}_j(X) \geq \mathcal{E}_j(X_0) \) for all \( j \) and \( \mathcal{D}_k(X) < \mathcal{D}_k(X_0) \) for all \( k \).
- if there is no vector \( v \) from the set
  \[ \mathcal{PP}S^R = \{ (\mathcal{D}_1(X), \ldots, \mathcal{D}_K(X)) : \mathcal{E}_j(X) \geq \mathcal{E}_j(X_0), j = 1, \ldots, J, X \in \mathcal{X} \} \]
  for which \( v \prec_{st} (\mathcal{D}_1(X_0), \ldots, \mathcal{D}_K(X_0)) \).
Model 1 - equivalent formulation

\[ \theta_1^l(X_0) = \min \max_{k=1,\ldots,K} \frac{D_k(X)}{D_k(X_0)} \]

s.t.

\[ \mathcal{E}_j(X) \geq \mathcal{E}_j(X_0), \ j = 1, \ldots, J, \]

\[ X \in \mathcal{X}. \]

\[ D(X) = \max_{k=1,\ldots,K} D_k(X)/D_k(X_0) \] defines a deviation measure.
For a benchmark $X_0 \in \mathcal{X}$, we can introduce a generalized model

$$\theta_2^I(X_0) = \min \frac{1}{K} \sum_{k=1}^{K} \theta_k$$

s.t.

$$\mathcal{E}_j(X) \geq \mathcal{E}_j(X_0), \ j = 1, \ldots, J,$$

$$\mathcal{D}_k(X) \leq \theta_k \cdot \mathcal{D}_k(X_0), \ k = 1, \ldots, K,$$

$$0 \leq \theta_k \leq 1, \ k = 1, \ldots, K,$$

$$X \in \mathcal{X}.$$
Set of efficient opportunities

Proposition

\[ X_0 \in \Psi_2 \]

- if for all \( X \in \mathcal{X} \) for which \( E_j(X) \geq E_j(X_0) \) for all \( j \) holds that \( D_k(X) \geq D_k(X_0) \) for all \( k \), i.e. there is no \( X \in \mathcal{X} \) for which \( E_j(X) \geq E_j(X_0) \) for all \( j \) and \( D_k(X) < D_k(X_0) \) for at least one \( k \).

- if there is no vector \( v \) from the set

\[ \mathcal{P} \mathcal{P} \mathcal{S}^R = \{(D_1(X), \ldots, D_K(X)) : E_j(X) \geq E_j(X_0), j = 1, \ldots, J, X \in \mathcal{X}\} \]

for which \( v \prec_w (D_1(X_0), \ldots, D_K(X_0)) \).
Model 2 - equivalent formulation

The model is obviously equivalent to

$$\theta_2^I(X_0) = \min \frac{1}{K} \sum_{k=1}^{K} \frac{D_k(X)}{D_k(X_0)}$$

s.t.

$$E_j(X) \geq E_j(X_0), \ j = 1, \ldots, J, \quad (7)$$
$$D_k(X) \leq D_k(X_0), \ k = 1, \ldots, K,$$
$$X \in \mathcal{X}.$$
Additive or slacks-based DEA model

\[ \theta_3^I(X_0) = \min \frac{1}{K} \sum_{k=1}^{K} \frac{D_k(X_0) - s^-_k}{D_k(X_0)} \]

s.t.

\[ E_j(X) \geq E_j(X_0), \quad j = 1, \ldots, J, \quad (8) \]

\[ D_k(X) + s^-_k = D_k(X_0), \quad k = 1, \ldots, K, \]

\[ s^-_k \geq 0, \quad k = 1, \ldots, K, \]

\[ X \in \mathcal{X}. \]
Proposition

The considered DEA models are units invariant.

For arbitrary $k$ and $j$ $\lambda D_k(X) = D_k(\lambda X)$ which implies $E_j(\lambda X) = \lambda E_j(\lambda X)$ for arbitrary $X \in \mathcal{X}$ and $\lambda > 0$. 
Proposition

Let $K \geq 2$. Then for a benchmark $X_0 \in \mathcal{X}$, the following relations between the optimal values (efficiency scores) hold

$$\theta^I_0(X_0) \geq \theta^I_1(X_0) \geq \theta^I_2(X_0) = \theta^I_3(X_0).$$

For the sets of efficient portfolios we obtain

$$\Psi^I_3 = \Psi^I_2 \subseteq \Psi^I_1 \subseteq \Psi^I_0.$$
Test strength

We can construct weaker tests if we restrict the number of investment opportunities in the portfolio leading to the models with the limited diversification.

Proposition

For $X^L_D$ and $X'^L_D$, with $# > #'$, it holds $\theta(X_0) \leq \theta'(X_0)$, where $\theta(X_0)$ and $\theta'(X_0)$ denote the efficiency scores with respect to the sets $X^L_D$, and $X'^L_D$ respectively.
Input-output oriented DC DEA models - (in)efficiency is measured also with respect to the outputs (assume $\varepsilon_j(X_0) > 0$):

- optimal values (efficiency scores) and strength can be compared,
- input and input-output oriented models can be compared: I-O tests are stronger in general,

We assume that $\mathcal{E}_j(X_0)$ is positive for at least one $j$. An input-output oriented test where inefficiency is measured with respect to the inputs and outputs separately can be formulated as follows

$$\theta_{1}^{I-O}(X_0) = \min_{\theta, \varphi, X} \frac{\theta}{\varphi} \quad \text{s.t.}$$

$$\mathcal{E}_j(X) \geq \varphi \cdot \mathcal{E}_j(X_0), \quad j = 1, \ldots, J,$$

$$\mathcal{D}_k(X) \leq \theta \cdot \mathcal{D}_k(X_0), \quad k = 1, \ldots, K,$$

$$0 \leq \theta \leq 1, \quad \varphi \geq 1,$$

$$X \in \mathcal{X}.$$
Input-output oriented test 1

Let $\mathcal{X} = \mathcal{X}^{FD}$. Setting $1/t = \varphi$ and substitute $\tilde{x}_i = tx_i$, $\tilde{\theta} = t\theta$, and $\tilde{\varphi} = t\varphi$, results into an input oriented DEA test with nonincreasing return to scale (NIRS):

$$\theta_{1\rightarrow O}^I(R_0) = \min_{\tilde{\theta}, \tilde{x}_i} \tilde{\theta}$$

s.t. $\mathcal{E}_j \left( \sum_{i=1}^{n} R_i \tilde{x}_i \right) \geq \mathcal{E}_j(R_0), \ j = 1, \ldots, J,$

$\mathcal{D}_k \left( \sum_{i=1}^{n} R_i \tilde{x}_i \right) \leq \tilde{\theta} \cdot \mathcal{D}_k(R_0), \ k = 1, \ldots, K,$

$$\sum_{i=1}^{n} \tilde{x}_i \leq 1, \ \tilde{x}_i \geq 0, \ 1 \geq \tilde{\theta} \geq 0.$$ 

Note that it is important for the reformulation that all inputs $\mathcal{D}_k$ and all outputs $\mathcal{E}_j$ are positively homogeneous. We obtained a convex programming problem.
Input-output oriented test 1

Proposition

$X_0 \in \Psi^{I-O}$:

- if there is no $X \in \mathcal{X}$ for which either $(\mathcal{E}_j(X) \geq \mathcal{E}_j(X_0)$ for all $j$ and $\mathcal{D}_k(X) < \mathcal{D}_k(X_0)$ for all $k$) or $(\mathcal{E}_j(X) > \mathcal{E}_j(X_0)$ for all $j$ and $\mathcal{D}_k(X) \leq \mathcal{D}_k(X_0)$ for all $k$), or equivalently

- if there is no vector $v$ from the production possibility set for which either

$$v \prec_{pst}\{1, \ldots, K\} (\mathcal{D}_1(X_0), \ldots, \mathcal{D}_K(X_0), -\mathcal{E}_1(X_0), \ldots, -\mathcal{E}_J(X_0)),$$

or

$$v \prec_{pst}\{K+1, \ldots, K+J\} (\mathcal{D}_1(X_0), \ldots, \mathcal{D}_K(X_0), -\mathcal{E}_1(X_0), \ldots, -\mathcal{E}_J(X_0)).$$
Input-output oriented test 2

For a benchmark $X_0 \in \mathcal{X}$ formulated as

$$
\theta_{2}^{I-O}(X_0) = \min_{\theta_k, \varphi_j, X} \frac{1}{K} \sum_{k=1}^{K} \theta_k \frac{1}{J} \sum_{j=1}^{J} \varphi_j \\
\text{s.t.}
\begin{align*}
\mathcal{E}_j(X) & \geq \varphi_j \cdot \mathcal{E}_j(X_0), & j = 1, \ldots, J, \quad (10) \\
\mathcal{D}_k(X) & \leq \theta_k \cdot \mathcal{D}_k(X_0), & k = 1, \ldots, K, \\
0 & \leq \theta_k \leq 1, & \varphi_k \geq 1, & k = 1, \ldots, K, \\
X & \in \mathcal{X}.
\end{align*}
$$
Input-output oriented test 2

Let $\mathcal{X} = \mathcal{X}^{FD}$. For a benchmark $R_0 \in \mathcal{X}$ formulated as

$$
\theta_2(R_0) = \min_{\bar{\theta}_k, \bar{\phi}_j, \bar{x}_i} \frac{1}{K} \sum_{k=1}^{K} \bar{\theta}_k
$$

s.t. $\frac{1}{J} \sum_{j=1}^{J} \bar{\phi}_j = 1,$

$$
\mathcal{E}_j \left( \sum_{i=1}^{n} R_i \bar{x}_i \right) \geq \bar{\phi}_j \cdot \mathcal{E}_j(R_0), \quad j = 1, \ldots, J,
$$

$$
\mathcal{D}_k \left( \sum_{i=1}^{n} R_i \bar{x}_i \right) \leq \bar{\theta}_k \cdot \mathcal{D}_k(R_0), \quad k = 1, \ldots, K,
$$

$$
\sum_{i=1}^{n} \bar{x}_i \leq 1, \quad \bar{x}_i \geq 0, \quad i = 1, \ldots, n,
$$

$$
\bar{\phi}_j \geq 0, \quad 1 \geq \bar{\theta}_k \geq 0.
$$
Proposition

\( X_0 \in \Psi^{I-O}_2 : \)

- if there is no \( X \in \mathcal{X} \) for which \( \mathcal{E}_j(X) \geq \mathcal{E}_j(X_0) \) for all \( j \) and \( \mathcal{D}_k(X) \leq \mathcal{D}_k(X_0) \) for all \( k \) with at least one inequality strict, or equivalently

- if there is no vector \( v \) from the production possibility set for which

\[
v \prec_{pw(\{1,\ldots,K+J\})} (\mathcal{D}_1(X_0), \ldots, \mathcal{D}_K(X_0), -\mathcal{E}_1(X_0), \ldots, -\mathcal{E}_J(X_0)).\]
Input-output oriented test 3

The additive or **slacks-based** test inspired by Tone (2001):

\[
\theta_{3}^{I-O}(X_0) = \min_{s^-_k, s^+_j, X} \frac{1}{K} \sum_{k=1}^{K} \frac{D_k(X_0) - s^-_k}{D_k(X_0)} \quad \text{s.t.} \\
\frac{1}{J} \sum_{j=1}^{J} \frac{E_j(X_0) + s^+_j}{E_j(X_0)} \\
E_j(X) - s^+_j \geq E_j(X_0), \ j = 1, \ldots, J, \\
D_k(X) + s^-_k \leq D_k(X_0), \ k = 1, \ldots, K, \\
s^+_j \geq 0, \ s^-_k \geq 0, \\
X \in \mathcal{X}.
\]

Thus, the score can be interpreted as the ratio of the mean input and the mean output inefficiencies. The objective function can be rewritten as

\[
1 - \frac{1}{K} \sum_{k=1}^{K} \frac{s^-_k}{D_k(X_0)} \\
1 + \frac{1}{J} \sum_{j=1}^{J} \frac{s^+_j}{E_j(X_0)}.
\]
Properties and relations

Proposition

Let $\max\{J, K\} \geq 2$. Then for a benchmark $X_0 \in \mathcal{X}$ with $D_k(X_0) > 0$ for all $k$ and $E_j(X_0) > 0$ for all $j$, the following relations hold

$$\theta^{I-O}_0(X_0) \geq \theta^{I-O}_1(X_0) \geq \theta^{I-O}_2(X_0) = \theta^{I-O}_3(X_0).$$

Then, for the sets of efficient portfolios it can be obtained

$$\psi^{I-O}_3 = \psi^{I-O}_2 \subseteq \psi^{I-O}_1 \subseteq \psi^{I-O}_0.$$
Properties and relations

Proposition

Let $\max\{J, K\} \geq 2$. Then for a benchmark $X_0 \in \mathcal{X}$ with $D_k(X_0) > 0$ for all $k$ and $E_j(X_0) > 0$ for all $j$, the following relations hold

$$\theta^I_1(X_0) \geq \theta^{I-O}_1(X_0), \quad \theta^I_2(X_0) \geq \theta^{I-O}_2(X_0).$$

Then, for the sets of efficient portfolios can be obtained

$$\Psi^{I-O}_1 \subseteq \Psi^I_1, \quad \Psi^{I-O}_2 \subseteq \Psi^I_2.$$
We consider CVaR deviations $\mathcal{D}_{\alpha_k}$ for $\alpha_k \in (0, 1)$, $k = 1, \ldots, K$ as the inputs and the expectation as the output, i.e. $J = 1$ and $\mathcal{E}_1(X) = \mathbb{E}X$:

$$
\min \theta \\
\text{s.t.} \\
\sum_{i=1}^{n} \mathbb{E}[R_i]x_i \geq \mathbb{E}[R_0], \\
\mathcal{D}_{\alpha_k}\left(\sum_{i=1}^{n} R_ix_i\right) \leq \theta \cdot \mathcal{D}_{\alpha_k}(R_0), \ k = 1, \ldots, K, \\
\sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \ i = 1, \ldots, n.
$$

(full diversification)
Mean absolute deviation from \((1 - \alpha)\)-th quantile

CVaR deviation

For any \(\alpha \in (0, 1)\) a finite, continuous, lower range dominated deviation measure

\[
D_\alpha(X) = CVaR_\alpha(X - \mathbb{E}[X]).
\] (12)

The deviation is also called **weighted mean absolute deviation from the** \((1 - \alpha)\)-th quantile, see Ogryczak, Ruszczynski (2002), because it can be expressed as

\[
D_\alpha(X) = \min_{\xi \in \mathbb{R}} \frac{1}{1 - \alpha} \mathbb{E}[\max\{(1 - \alpha)(X - \xi), \alpha(\xi - X)\}]
\] (13)

with the minimum attained at any \((1 - \alpha)\)-th quantile. In relation with CVaR minimization formula, see Pflug (2000), Rockafellar and Uryasev (2000, 2002).
DC DEA model with CVaR deviations
Input oriented

For discretely distributed returns \((r_{is}, s = 1, \ldots, S, p_s = 1/S)\) LP:

\[
\theta_1^I(R_0) = \min \theta \quad \text{s.t.} \quad \sum_{i=1}^{n} \mathbb{E}[R_i] x_i \geq \mathbb{E}[R_0],
\]

\[
\frac{1}{S} \sum_{s=1}^{S} u_{sk} \leq \theta \cdot \mathcal{D}_{\alpha_k}(R_0), \quad k = 1, \ldots, K,
\]

\[
u_{sk} \geq \sum_{i=1}^{n} x_i r_{is} - \xi, \quad s = 1, \ldots, S, \quad k = 1, \ldots, K,
\]

\[
u_{sk} \geq \frac{\alpha_k}{1 - \alpha_k} (\xi - \sum_{i=1}^{n} x_i r_{is}), \quad s = 1, \ldots, S, \quad k = 1, \ldots, K,
\]

\[
\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \quad i = 1, \ldots, n.
\]
DC DEA model with CVaR deviations

For discretely distributed returns \((r_{is}, s = 1, \ldots, S, p_s = 1/S)\) LP:

\[
\theta_1^{I-O}(R_0) = \min_{\theta, x_i, u_{sk}, \xi_k} \theta \quad \text{s.t.} \quad \sum_{i=1}^{n} \mathbb{E}[R_i]x_i \geq \mathbb{E}[R_0],
\]

\[
\frac{1}{S} \sum_{s=1}^{S} u_{sk} \leq \theta \cdot D_{\alpha_k}^{S}(R_0), \quad k = 1, \ldots, K,
\]

\[
u_{sk} \geq \left( \sum_{i=1}^{n} x_i r_{is} - \xi_k \right), \quad s = 1, \ldots, S, k = 1, \ldots, K,
\]

\[
u_{sk} \geq \frac{\alpha_k}{1 - \alpha_k} \left( \xi_k - \sum_{i=1}^{n} x_i r_{is} \right),
\]

\[
\sum_{i=1}^{n} x_i \leq 1, \quad x_i \geq 0, \quad i = 1, \ldots, n.
\]
Financial indices

We consider the following 25 world financial indices which are listed on Yahoo Finance:

- **America** (5): MERVAL BUENOS AIRES, IBOVESPA, S&P TSX Composite index, S&P 500 INDEX RTH, IPC,
- **Asia/Pacific** (11): ALL ORDINARIES, SSE Composite Index, HANG SENG INDEX, BSE SENSEX, Jakarta Composite Index, FTSE Bursa Malaysia KLCI, NIKKEI 225, NZX 50 INDEX GROSS, STRAITS TIMES INDEX, KOSPI Composite Index, TSEC weighted index,
- **Europe** (8): ATX, CAC 4, DAX, AEX, SMI, OMX Stockholm PI, SMI, FTSE 100,
- **Middle East** (1): TEL AVIV TA-100 IND.

The same dataset analyzed by Branda and Kopa (2010, 2012).
In our analysis we describe each index by its weekly rates of returns. We divided the returns into three datasets:

- **before crises** (B): September 11, 2006 - September 15, 2008
- **during crises** (D): September 16, 2008 - September 20, 2010
- **whole period** (W).

CVaR deviation levels: $\alpha_k \in \{0.75, 0.9, 0.95, 0.99\}$

- DEA optimal values/scores with $\# = n$...
- DEA optimal values/scores with $\# = 2$...
- DEA optimal values/scores with $\# = 1$...
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<th>( D )</th>
<th>( W )</th>
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On relations between DEA and stochastic dominance efficiency

Contents

1. Efficiency of investment opportunities
2. Data Envelopment Analysis
3. Diversification-consistent DEA based on general deviation measures
   - General deviation measures
   - Diversification-consistent DEA models
   - Financial indices efficiency – empirical study
4. On relations between DEA and stochastic dominance efficiency
   - Second Order Stochastic Dominance
   - Data Envelopment Analysis
   - Numerical comparison
5. References
Data envelopment analysis (production theory, returns to scale, radial/slacks-based/directional distance models, primal/dual formulations, multiobjective opt. – Pareto efficiency, chance constraints),

large literature: handbooks on DEA, Omega, EJOR, JORS, ...

Stochastic dominance efficiency (pairwise, convex, portfolio): Is there \( R \) such that

\[ R \succeq_{SSD} R_0? \]

No – \( R_0 \) is efficient.

Compare with the problem

\[
\max f(R) : \text{s.t. } R \succeq_{SSD} R_0.
\]
Second order stochastic dominance

$F_1, F_2...$ cumulative probability distributions functions of random variables $X_1, X_2$.

**Second order (strict) stochastic dominance (SSD):**

$X_1 \succ_{SSD} X_2$ iff $\mathbb{E}_{F_1} u(x) - \mathbb{E}_{F_2} u(x) \geq 0$ for every concave utility function $u$ with at least one strict inequality.

Consider twice cumulated probability distributions functions:

$$F_i^{(2)}(t) = \int_{-\infty}^{t} F_i(x) \, dx \quad i = 1, 2.$$

**Theorem** (Hanoch & Levy (1969)):

$X_1 \succ_{SSD} X_2 \iff F_1^{(2)}(t) \leq F_2^{(2)}(t) \quad \forall t \in \mathbb{R}$ with at least one strict inequality.
We consider

- $n$ assets and we denote $R_i$ the rate of return of $i$-th asset with finite mean value, $\mathbf{r} = \{R_1, \ldots, R_n\}$.
- a **discrete probability distribution** of rate of returns described by scenarios $r_{i,s}$, $s = 1, \ldots, S$ that are taken with **equal probabilities** $p_s = 1/S$.
- a decision maker that may combine the assets into portfolios represented by weights $\mathbf{x} = \{x_1, \ldots, x_n\}$.
- the set of feasible weights (no short sales allowed):

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R} | \sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, \ldots, n\}. \quad (14)$$
A given portfolio $\tau \in \mathcal{X}$ is **SSD portfolio efficient** if and only if there exists no portfolio $\lambda \in \mathcal{X}$ such that $r'\lambda \succ_{SSD} r'\tau$. Otherwise, portfolio $\tau$ is SSD inefficient.

**SSD portfolio efficiency tests**: Post (2003), Kuosmanen (2004), Kopa and Chovanec (2008)...
Fishburn (1974) defines a concept of **convex stochastic dominance**: We say that portfolio $\mathbf{x}$ is convex SSD inefficient if every investor prefers some of the assets to portfolio $\mathbf{x}$.

**Formal definition:**
A given portfolio $\mathbf{r}$ is **convex SSD efficient** if there exists at least some nondecreasing concave $u$ such that $\mathbb{E}u(r') > \mathbb{E}u(r_i)$ for all $i = 1, 2, ..., n$. 
Convex SSD efficiency test

Bawa et al. (1985): Let \( D = \{ d_1, d_2, ..., d_{(n+1)S} \} \) be the set of all scenario returns of the assets and portfolio \( x \), that is, for every \( i \in \{1, 2, ..., n+1\} \) and \( s \in \{1, 2, ..., S\} \) exists \( k \in \{1, 2, ..., (n+1)S\} \) such that \( r_{i,s} = d_k \) and vice versa, where \( r_{(n+1),s} = \sum_{i=1}^{n} r_{i,s}x_i \).

Convex SSD efficiency test of portfolio \( x \):

\[
\delta^*(x) = \max_{\delta_k, x_i} \sum_{k=1}^{(n+1)S} \delta_k
\]

s.t. \( F_x^{(2)}(d_k) - \sum_{i=1}^{n} \bar{x}_i F_i^{(2)}(d_k) \geq \delta_k, \ k = 1, 2, ..., (n+1)S, \)

\[
\delta_k \geq 0, \ k = 1, 2, ..., (n+1)S,
\]

\( \bar{x} \in \mathcal{X} \).

A given portfolio \( x \) is convex SSD inefficient if \( \delta^*(x) \) given by (15) is strictly positive. Otherwise, portfolio \( x \) is convex SSD efficient.
Data Envelopment Analysis (DEA)

Charnes, Cooper and Rhodes (1978): a way how to state efficiency of a decision making unit over all other decision making units with the same structure of inputs and outputs.

Let $Z_{1i}, \ldots, Z_{Ki}$ denote the inputs and $Y_{1i}, \ldots, Y_{Ji}$ denote the outputs of the unit $i$ from $n$ considered units. DEA efficiency of the unit $0 \in \{1, \ldots, n\}$ is then evaluated using the optimal value of the following program where weighted inputs are compared with the weighted outputs.

“Are inputs transformed into outputs in an efficient way?”
Banker, Charnes and Cooper (1984): DEA model with Variable Returns to Scale:

$$
\begin{align*}
\min \theta \\
\text{s.t.} \\
\sum_{i=1}^{n} x_i Y_{ji} & \geq Y_{j0}, \ j = 1, \ldots, J, \\
\sum_{i=1}^{n} x_i Z_{ki} & \leq \theta \cdot Z_{k0}, \ k = 1, \ldots, K, \\
\sum_{i=1}^{n} x_i & = 1, \ x_i \geq 0, \ i = 1, \ldots, n.
\end{align*}
$$
Efficiency of mutual funds or financial indexes:

- Basso and Funari (2001, 2003): standard deviation and semideviations, beta coefficient, costs as the inputs, expected return or expected excess return, ethical measure and stochastic dominance criterion as the outputs.
- B. and Kopa (2010, 2012A): VaR, CVaR, sd, lsd, drawdown measures (DaR, CDaR) as the inputs, gross return as the output; comparison with SSD.
Equivalent DEA test to convex SSD efficiency test

- Let \( \tilde{D} = \{d_1, d_2, \ldots, d_{(n+1)S}\} \) be the set of all sorted scenario returns of the assets \( R_i \) and portfolio \( x \).
- The test can be rewritten using lower partial moments \( L_i(d) = 1/S \sum_{s=1}^{S} [d - r_{i,s}]^+ \).
- Based on the results proposed by Bawa et al. (1985)
B. and Kopa (2013): Benchmark portfolio $\mathbf{x}$ with return $R_0 = \sum_{i=1}^{n} R_i x_i$. Find the index $\tilde{k} = \arg\min\{ k : L_0(d_k) > 0 \}$. Then the DEA-risk model with **variable return to scale** and $K = (n + 1)S - 1$ inputs

$$\delta^{CDEA}(R_0) = \min_{\bar{x}_i,\varphi,\theta_k} \frac{1}{K - \tilde{k} + 2} \left[ \sum_{k=\tilde{k}}^{K} \theta_k + \frac{1}{\varphi} \right]$$

$$\sum_{i=1}^{n} \bar{x}_i \mathbb{E}[R_i] \geq \varphi \cdot \mathbb{E}[R_0],$$

$$\sum_{i=1}^{n} \bar{x}_i L_i(d_k) \leq \theta_k \cdot L_0(d_k), \; k = 1, \ldots, K,$$

$$0 \leq \theta_k \leq 1, \; \varphi \geq 1,$$

$$\sum_{i=1}^{n} \bar{x}_i = 1, \; \bar{x}_i \geq 0, \; i = 1, \ldots, n.$$

**is equivalent** to convex SSD efficiency test.
Diversification-consistent DEA tests
Input oriented

Recently, general DEA tests with diversification effect were introduced by Lamb and Tee (2012) for a benchmark with return $R_0$:

$$\theta^{DC}(R_0) = \min_{\theta, x_i} \theta$$

$$-\text{CVaR}_{\varepsilon_j} \left( - \sum_{i=1}^{n} R_i x_i \right) \geq -\text{CVaR}_{\varepsilon_j}(-R_0), \ j = 1, \ldots, J, \quad (17)$$

$$\text{CVaR}_{\alpha_k}^+ \left( - \sum_{i=1}^{n} R_i x_i \right) \leq \theta \cdot \text{CVaR}_{\alpha_k}^+(-R_0), \ k = 1, \ldots, K,$$

$$\sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \ i = 1, \ldots, n,$$

where $\text{CVaR}_{\alpha_k}^+ = \max\{\text{CVaR}_{\alpha}, 0\}$, and $\alpha_k, \varepsilon_j$ are different levels, the positive parts of CVaRs serve as the inputs and expected return as the output.
Diversification-consistent test
Input-output oriented I

B. and Kopa (2013): Let \( \text{CVaR}_\alpha(-R_0) > 0 \) for \( \alpha \in \{\alpha_1, \ldots, \alpha_K\} \) and \( \text{CVaR}_\varepsilon(-R_0) < 0 \) for \( \varepsilon \in \{\varepsilon_1, \ldots, \varepsilon_J\} \)

\[
\theta^{DC} I-O I(R_0) = \min_{\theta, \varphi, x_i} \frac{1}{K + J} \left[ \theta + \frac{1}{\varphi} \right]
\]

\(-\text{CVaR}_{\varepsilon_j} \left( - \sum_{i=1}^{n} R_i x_i \right) \geq \varphi \cdot (-\text{CVaR}_{\varepsilon_j}(-R_0)), \ j = 1, \ldots, J, (18)\)

\(\text{CVaR}_{\alpha_k} \left( - \sum_{i=1}^{n} R_i x_i \right) \leq \theta \cdot \text{CVaR}_{\alpha_k}(-R_0), \ k = 1, \ldots, K,\)

\(0 \leq \theta \leq 1, \ \varphi \geq 1,\)

\(\sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \ i = 1, \ldots, n.\)

Note that \(\text{CVaR}_0(-R_0) = \mathbb{E}[-R_0]\), i.e. expected loss can be also included into this model without any changes in its formulation.
Diversification-consistent test
Input-output oriented II

B. and Kopa (2013):

\[
\theta^{DC\mbox{ I-O II}}(R_0) = \min_{\theta_k, \varphi_j, x_i} \frac{1}{K + J} \left[ \sum_{k=1}^{K} \theta_k + \sum_{j=1}^{J} \frac{1}{\varphi_j} \right]
\]

\[-\text{CVaR}_{\varepsilon_j} \left( - \sum_{i=1}^{n} R_i x_i \right) \geq \varphi_j \cdot \left( -\text{CVaR}_{\varepsilon_j}(-R_0) \right), \quad j = 1, \ldots, J, (19)\]

\[\text{CVaR}_{\alpha_k} \left( - \sum_{i=1}^{n} R_i x_i \right) \leq \theta_k \cdot \text{CVaR}_{\alpha_k}(-R_0), \quad k = 1, \ldots, K,\]

\[0 \leq \theta_k \leq 1, \quad \varphi_j \geq 1,\]

\[\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \quad i = 1, \ldots, n.\]

These DEA-risk models can be seen as the extension of Russel measure DEA model (see Cook and Seiford (2009)).
B. and Kopa (2013): We assume that no $\text{CVaR}_{\alpha_s}$ of the benchmark is equal to zero for $\alpha_s = s/S$, $s \in \Gamma = \{0, 1, \ldots, S - 1\}$

- Let $\tilde{s} = \arg \max \{ s \in \Gamma : \text{CVaR}_{\alpha_s}(-R_0) < 0 \}$.
- $\varepsilon_j \in \{0/S, 1/S, \ldots, \tilde{s}/S\}$, $J = \tilde{s} + 1$
- $\alpha_k \in \{(	ilde{s} + 1)/S, \ldots, (S - 1)/S\}$, $K = S - \tilde{s} - 1$.

Then the corresponding diversification-consistent DEA-risk model (I-O II) is equivalent to SSD portfolio efficiency test, that is, a benchmark $R_0 = \sum_{i=1}^{n} R_i x_i$ is DEA-risk efficient if and only if portfolio $x$ is SSD portfolio efficient.

B. and Kopa (2012B,2013): To compare the power of considered efficiency tests, we consider

- historical US stock market data, monthly excess returns from January 1982 to December 2011 (360 observations) of 48 representative industry stock portfolios that serve as the base assets. The industry portfolios are based on four-digit SIC codes and are from Kenneth French library.

- five DEA-risk models where CVaRs at levels $\alpha = 0.5, 0.75, 0.9, 0.95, 0.99, 0.995$ are used as the inputs and the expected return (the most commonly used reward measure) as the output.
## Numerical comparison

<table>
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Conclusions

- Standard DEA tests can be equivalent to convex stochastic dominance tests.
- DEA tests equivalent to portfolio stochastic dominance efficiency should take into account diversification effect leading diversification-consistent DEA tests.
1 Efficiency of investment opportunities

2 Data Envelopment Analysis

3 Diversification-consistent DEA based on general deviation measures
   - General deviation measures
   - Diversification-consistent DEA models
   - Financial indices efficiency – empirical study

4 On relations between DEA and stochastic dominance efficiency
   - Second Order Stochastic Dominance
   - Data Envelopment Analysis
   - Numerical comparison

5 References


Thank you for your attention.

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