Copula Functions in Credit Risk Modeling

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Contents

1. Duo Basket Example
2. Copula Functions
3. Portfolio Credit Risk
4. Tail Dependence and Limit Behavior
1 Duo Basket Example

2 Copula Functions

3 Portfolio Credit Risk

4 Tail Dependence and Limit Behavior
This example is taken from Bluhm and Overbeck (2006)

Suppose we have two loans in our basket: loan A and loan B

For a one-year horizon we introduce Bernoulli variables $L_A$ and $L_B$

\[ L_A = \begin{cases} 1 & \text{if } CWI_A < c_A \\ 0 & \text{otherwise} \end{cases} \quad (1) \]
\[ L_B = \begin{cases} 1 & \text{if } CWI_B < c_B \\ 0 & \text{otherwise} \end{cases} \quad (2) \]

where $CWI_A$ and $CWI_B$ are random variables with correlation coefficient $\rho$, and $c_A$ and $c_B$ are real bounds.

Then we get for the Probabilities of Default the relations

\[ p_A = \mathbb{P}(L_A = 1) = \mathbb{P}(CWI_A < c_A) = F_A(c_A) \]
\[ p_B = \mathbb{P}(L_B = 1) = \mathbb{P}(CWI_B < c_B) = F_B(c_B) \]
Suppose that CWI\textsubscript{A} and CWI\textsubscript{B} are defined as follows:

\[
\begin{align*}
\text{CWI}_A &= \sqrt{\rho} \ Y + \sqrt{1 - \rho} \ \varepsilon_A \quad (3) \\
\text{CWI}_B &= \sqrt{\rho} \ Y + \sqrt{1 - \rho} \ \varepsilon_B \quad (4)
\end{align*}
\]

where \( Y, \varepsilon_A \) and \( \varepsilon_B \) are independent, standard normal random variables.
The first-to-default probability is defined as the probability that at least one obligor defaults, i.e.

\[ p_{1st} = \mathbb{P} \left( \{ \text{CWI}_A < c_A \} \cup \{ \text{CWI}_B < c_B \} \right) \]  \hspace{1cm} (5)

The second-to-default probability is defined as the probability that both obligors default, i.e.

\[ p_{2nd} = \mathbb{P} \left( \{ \text{CWI}_A < c_A \} \cap \{ \text{CWI}_B < c_B \} \right) \]  \hspace{1cm} (6)
Lemma 1

Suppose the CreditMetrics model given by (3) and (4). Then random variables $L_A$ and $L_B$ defined in (1) and (2) are conditionally independent for the given state of economy $Y = y$, and the conditional Probabilities of Default $g_{p_A,\rho}(y) = \mathbb{P}(L_A = 1|Y = y)$ and $g_{p_B,\rho}(y) = \mathbb{P}(L_B = 1|Y = y)$ are given by

\[ g_{p_A,\rho}(y) = \Phi \left( \frac{\Phi^{-1}(p_A) - \sqrt{\rho} y}{\sqrt{1 - \rho}} \right), \quad (7) \]

\[ g_{p_B,\rho}(y) = \Phi \left( \frac{\Phi^{-1}(p_B) - \sqrt{\rho} y}{\sqrt{1 - \rho}} \right). \quad (8) \]
Conditional independence of $L_A$ and $L_B$ follows from conditional independence of $CWI_A$ and $CWI_B$ defined in (3) and (4), where $\varepsilon_A$ and $\varepsilon_B$ are independent. The conditional Probability of Default for loan A can be computed as

\[
g_{p_A,\rho}(y) = \mathbb{P}(L_A = 1 | Y = y) = \mathbb{P}(CWI_A < c_A | Y = y) = \\
= \mathbb{P}\left(\sqrt{\rho} Y + \sqrt{1 - \rho} \varepsilon_A < c_A | Y = y\right) = \\
= \mathbb{P}\left(\varepsilon_A < \frac{c_A - \sqrt{\rho} Y}{\sqrt{1 - \rho}} | Y = y\right) = \\
= \Phi\left(\frac{c_A - \sqrt{\rho} y}{\sqrt{1 - \rho}}\right) = \\
= \Phi\left(\frac{\Phi^{-1}(p_A) - \sqrt{\rho} y}{\sqrt{1 - \rho}}\right).
\]

Then $g_{p_B,\rho}$ is computed analogously.
First-to-Default Probability

Proposition 1

The one-year first-to-default probability equals

\[
p_{1st} = \int_{-\infty}^{\infty} \left( g_{pA,\rho}(y) + (1 - g_{pA,\rho}(y)) g_{pB,\rho}(y) \right) d\Phi(y). \tag{9}
\]

where \( g_{pA,\rho} \) and \( g_{pB,\rho} \) are the one-year conditional Probabilities of Default, defined in (7) and (8).
First-to-Default Probability

Proof.

Denote $A = \{\text{CWI}_A < c_A\}$ and $B = \{\text{CWI}_B < c_B\}$. Due to conditional independence from Lemma 1 we can write the one-year first-to-default probability from the definition (5) as

$$p_{1st} = \mathbb{P}(A \cup B) =$$

$$= \mathbb{P}(A \cup (B \setminus A)) =$$

$$= \int_{-\infty}^{\infty} \mathbb{P}(A \cup (B \setminus A) \mid Y = y) \, d\Phi(y) =$$

$$= \int_{-\infty}^{\infty} \left( \mathbb{P}(A \mid Y = y) + \mathbb{P}\left((B \cap A^c) \mid Y = y\right) \right) \, d\Phi(y) =$$

$$= \int_{-\infty}^{\infty} \left( g_{PA,\rho}(y) + (1 - g_{PA,\rho}(y)) \, g_{PB,\rho}(y) \right) \, d\Phi(y).$$
Proposition 2

The one-year second-to-default probability equals

\[ p_{2nd} = \int_{-\infty}^{\infty} g_{pA,\rho}(y)g_{pB,\rho}(y) \, d\Phi(y). \]  

(10)

where \( g_{pA,\rho} \) and \( g_{pB,\rho} \) are the one-year conditional Probabilities of Default, defined in (7) and (8).
Second-to-Default Probability

Proof.

Again denote $A = \{\text{CWI}_A < c_A\}$ and $B = \{\text{CWI}_B < c_B\}$. Then due to conditional independence from Lemma 1 we can write the one-year second-to-default probability from the definition (6) as

$$p_{2nd} = \mathbb{P}(A \cap B) =$$

$$= \int_{-\infty}^{\infty} \mathbb{P}(A \cap B | Y = y) \, d\Phi(y) =$$

$$= \int_{-\infty}^{\infty} \mathbb{P}(A | Y = y) \, \mathbb{P}(B | Y = y) \, d\Phi(y) =$$

$$= \int_{-\infty}^{\infty} g_{\rho_A,\rho}(y) g_{\rho_B,\rho}(y) \, d\Phi(y).$$
Proposition 3

From definition we can write the one-year second-to-default probability in the equivalent form

\[ p_{2nd} = \Phi_{2,\rho} \left( \Phi^{-1}(p_A), \Phi^{-1}(p_B) \right) , \]  

(11)

where \( \Phi_{2,\rho} \) denotes the standard bivariate normal distribution function with correlation \( \rho \), i.e.

\[ \Phi_{2,\rho}(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left( -\frac{x^2 + y^2 - \rho xy}{2(1 - \rho^2)} \right) \, dx \, dy. \]  

(12)
Using the definition of the second-to-default probability and the joint distribution of \((\text{CWI}_A, \text{CWI}_B)'\), we can write

\[
p_{2nd} = \mathbb{P}\left(\{\text{CWI}_A < c_A\} \cap \{\text{CWI}_B < c_B\}\right) = \\
= \mathbb{P}\left(\text{CWI}_A < \Phi^{-1}(p_A), \text{CWI}_B < \Phi^{-1}(p_B)\right) = \\
= \Phi_{2,\rho}\left(\Phi^{-1}(p_A), \Phi^{-1}(p_B)\right).
\]
Proof.

Using the definition of the second-to-default probability and the joint distribution of \((\text{CWI}_A, \text{CWI}_B)\)' we can write

\[
p_{2nd} = P(\{\text{CWI}_A < c_A\} \cap \{\text{CWI}_B < c_B\}) =
= P\left(\text{CWI}_A < \Phi^{-1}(p_A), \text{CWI}_B < \Phi^{-1}(p_B)\right) = \\
= \Phi_{2,\rho}\left(\Phi^{-1}(p_A), \Phi^{-1}(p_B)\right).
\]
Question

What is the joint distribution of CWI_A and CWI_B given (3) and (4)?

Recall: Y, ε_A and ε_B are independent, standard normal random variables

\[
\begin{align*}
\text{CWI}_A &= \sqrt{\rho} \cdot Y + \sqrt{1 - \rho} \cdot \varepsilon_A \\
\text{CWI}_B &= \sqrt{\rho} \cdot Y + \sqrt{1 - \rho} \cdot \varepsilon_B
\end{align*}
\]
Joint Distribution

Proposition 4

Let $Y$, $\varepsilon_A$, and $\varepsilon_B$ be independent, standard normal random variables. Then $\text{CWI}_A$ and $\text{CWI}_B$ defined in (3) and (4) have the bivariate normal joint distribution

$$
\begin{pmatrix}
\text{CWI}_A \\
\text{CWI}_B
\end{pmatrix}
\sim
\mathcal{N}_2
\left(
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 & \rho \\
\rho & 1
\end{pmatrix}
\right).
$$

(13)
Proof.

If we rewrite equations (3) and (4) into a matrix form, we get

\[
\begin{pmatrix}
  \text{CWI}_A \\
  \text{CWI}_B
\end{pmatrix}
= \begin{pmatrix}
  \sqrt{\rho} & \sqrt{1-\rho} & 0 \\
  \sqrt{\rho} & 0 & \sqrt{1-\rho}
\end{pmatrix}
\begin{pmatrix}
  Y \\
  \varepsilon_A \\
  \varepsilon_B
\end{pmatrix}
= BZ,
\] (14)

where due to the independence of its components the random vector

\[Z = (Y, \varepsilon_A, \varepsilon_B)'
\]

has the joint standard normal distribution.

Then \[\text{CWI}_A = BZ\] has also a normal distribution with expectation and variance matrix:

\[
\mathbb{E}
\begin{pmatrix}
  \text{CWI}_A \\
  \text{CWI}_B
\end{pmatrix}
= B \mathbb{E} Z = \begin{pmatrix}
  0 \\
  0
\end{pmatrix},
\]

\[
\text{Var}
\begin{pmatrix}
  \text{CWI}_A \\
  \text{CWI}_B
\end{pmatrix}
= \text{Var}(BZ) = B(\text{Var} Z)B' = BB' = \begin{pmatrix}
  1 & \rho \\
  \rho & 1
\end{pmatrix}.
\]
The joint default probability

\[ p_{2nd} = \Phi_{2,\rho} \left( \Phi^{-1}(p_A), \Phi^{-1}(p_B) \right) \]

In the terms of a Gaussian copula we can rewrite the joint default probability in the form

\[ p_{2nd} = C_{2,\rho} \left( p_A, p_B \right) \]
**Definition**

**Definition 1**

A *copula* is the distribution function of a random vector in $\mathbb{R}^d$ with standard uniform marginal distributions.

**Definition 2**

A *copula* is any function $C : [0, 1]^d \rightarrow [0, 1]$ which has the three properties:

1. $C(x_1, \ldots, x_d)$ is increasing in each component $x_i$.
2. $C(1, \ldots, 1, x_i, 1, \ldots, 1) = x_i$ for all $i \in \{1, \ldots, d\}$, $x_i \in [0, 1]$.
3. For all $(a_1, \ldots, a_d), (b_1, \ldots, b_d) \in [0, 1]^d$ with $a_i \leq b_i$ we have:

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(x_{i_1}, \ldots, x_{d_{i_d}}) \geq 0, \quad (15)$$

where $x_{j_1} = a_j$ and $x_{j_2} = b_j$ for all $j \in \{1, \ldots, d\}$. 

Michal Rychnovský

Copula Functions in Credit Risk Modeling
Frechét-Hoeffding’s bounds

**Theorem 1**

*(Frechét-Hoeffding’s bounds)*

For any copula $C(u_1, \ldots, u_d)$ we have the bounds

\[
\max \left[ \sum_{i=1}^{d} u_i + 1 - d, 0 \right] \leq C(u_1, \ldots, u_d) \leq \min[u_1, \ldots, u_d]. \tag{16}
\]
Proof.

Let \( U_1, \ldots, U_d \sim U(0, 1) \). The first inequality is given by

\[
C(u_1, \ldots, u_d) = \mathbb{P}\left( \bigcap_{i=1}^{d} \{ U_i \leq u_i \} \right) = 1 - \mathbb{P}\left( \bigcup_{i=1}^{d} \{ U_i > u_i \} \right) \geq 1 - \sum_{i=1}^{d} \mathbb{P}(U_i > u_i) = \sum_{i=1}^{d} u_i + 1 - d.
\]

And the second inequality follows from the expression

\[
C(u_1, \ldots, u_d) = \mathbb{P}\left( \bigcap_{i=1}^{d} \{ U_i \leq u_i \} \right) \leq \mathbb{P}(U_k \leq u_k) \quad \text{for all} \quad k \in \{1, \ldots, d\}.
\]
A copula $C$ is called **exchangeable** if it is the distribution function of an exchangeable random vector of standard uniform variates, i.e. if

$$C(u_1, \ldots, u_d) = C(u_{\Pi(1)}, \ldots, u_{\Pi(d)})$$

for any permutation $(\Pi(1), \ldots, \Pi(d))$ of $(1, \ldots, d)$. 

Proposition First

Proposition 5

Let $F$ be a distribution function and $F^{-1}$ denote the quantile function, i.e. $F^{-1}(u) = \inf\{x : F(x) \geq u\}$. The following two properties hold.

1. (Quantile transformation) If $U$ has the standard uniform distribution, then $F^{-1}(U)$ has distribution function $F$.

2. (Probability transformation) If $Y$ has a continuous distribution function $F$, then $F(Y)$ has the standard uniform distribution.
Proposition First

Proof.

For all \( y \in \mathbb{R} \) and \( u \in [0, 1] \), we have

1. As \( F \) is increasing, we get

\[
\mathbb{P} \left( F^{-1}(U) \leq y \right) = \mathbb{P} \left( U \leq F(y) \right) = F(y).
\]

2. As \( F \) is continuous, \( F^{-1} \) is strictly increasing and

\[
\mathbb{P} \left( F(Y) \leq u \right) = \mathbb{P} \left( Y \leq F^{-1}(u) \right) = F(F^{-1}(u)) = u.
\]
Theorem 2

(Sklar 1959)
Let $F$ be a joint distribution function with margins $F_1, \ldots, F_d$. Then there exists a copula $C : [0, 1]^d \rightarrow [0, 1]$ such that, for all $x_1, \ldots, x_d$ in $\mathbb{R}^* = [-\infty, \infty]$, $F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$. \hspace{1cm} (17)

If the margins are continuous, then $C$ is unique. Conversely, if $C$ is a copula and $F_1, \ldots, F_d$ are univariate distribution functions, then the function $F$ defined in (17) is a joint distribution function with margins $F_1, \ldots, F_d$. 

Michal Rychnovský
Copula Functions in Credit Risk Modeling
Sklar’s Theorem

**Sketch of proof (first part).**

We show the existence and uniqueness of the copula for the case of continuous margins $F_1, \ldots, F_d$. The complete proof can be found for instance in Nelsen (1999).

For all $x_1, \ldots, x_d$ in $\mathbb{R}^* = [-\infty, \infty]$ we have

$$F(x_1, \ldots, x_d) = \mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d) =$$

$$= \mathbb{P}(F_1(X_1) \leq F_1(x_1), \ldots, F_d(X_d) \leq F_d(x_d)) =$$

$$= C(F_1(x_1), \ldots, F_d(x_d)),$$

where $C$ is a copula because $F_1(X_1), \ldots, F_d(X_d) \sim U(0, 1)$ according to Proposition 5 for continuous margins. If we now substitute $x_i = F_i^{-1}(u_i)$ for $u_i \in [0, 1], i \in \{1, \ldots, d\}$ to (17), we get a unique expression of the copula,

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)). \quad (18)$$
Sklar’s Theorem

Sketch of proof (second part).

Conversely, for a copula $C$ and univariate distribution functions $F_1, \ldots, F_d$, we construct a random vector $(X_1, \ldots, X_d) = (F_1^{-1}(U_1), \ldots, F_d^{-1}(U_d))$, where $U_1, \ldots, U_d$ are standard uniform random variables with joint distribution function $C$. Then for the distribution function $F$ of the random vector $(X_1, \ldots, X_d)$ we get

$$F(x_1, \ldots, x_d) = \mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d) = \mathbb{P}(F_1^{-1}(U_1) \leq x_1, \ldots, F_d^{-1}(U_d) \leq x_d) = \mathbb{P}(U_1 \leq F_1(x_1), \ldots, U_d \leq F_d(x_d)) = C(F_1(x_1), \ldots, F_d(x_d)).$$
Some Examples of Copulae

- **Independence copula**
  \[ C_\perp(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i \] (19)

- **Comonotonic copula**
  \[ C_\diamond(u_1, \ldots, u_d) = \min\{u_1, \ldots, u_d\} \] (20)

- **Gaussian copula**
  \[ C_{d,\Gamma}(u_1, \ldots, u_d) = \Phi_{d,\Gamma}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)) \] (21)

- **Student-t copula**
  \[ C_{d,\Gamma,m}(u_1, \ldots, u_d) = \Theta_{d,\Gamma,m}(\Theta^{-1}_m(u_1), \ldots, \Theta^{-1}_m(u_d)) \] (22)
Some Examples of Copulae

- Let \( \varphi : [0, 1] \rightarrow [0, \infty] \) be a continuous, strictly decreasing, convex function satisfying \( \varphi(1) = 0 \) and \( \varphi(0) = \infty \)

- Archimedean copula

  \[
  C_\varphi(u_1, \ldots, u_d) = \varphi^{-1}(\varphi(u_1) + \cdots + \varphi(u_d)) \quad (23)
  \]

- Gumbel copula

  \[
  \varphi_{Gu(\theta)}(x) = (-\log x)^\theta, \quad \theta \in [1, \infty)
  \]

  \[
  C_{\varphi_{Gu(\theta)}}(u_1, \ldots, u_d) = \exp \left( -\left((-\log u_1)^\theta + \cdots + (-\log u_d)^\theta\right)^{\frac{1}{\theta}} \right) \quad (24)
  \]

- Clayton copula

  \[
  \varphi_{Cl(\eta)}(x) = x^{-\eta} - 1, \quad \eta \in (0, \infty)
  \]

  \[
  C_{\varphi_{Cl(\eta)}}(u_1, \ldots, u_d) = \left(u_1^{-\eta} + \cdots + u_d^{-\eta} + 1 - d\right)^{\frac{1}{\eta}} \quad (25)
  \]
Duo Basket Example
Copula Functions
Portfolio Credit Risk
Tail Dependence and Limit Behavior

Contents

1. Duo Basket Example
2. Copula Functions
3. Portfolio Credit Risk
4. Tail Dependence and Limit Behavior
This section is based on McNeil et al. (2005)

Let $X = (X_1, \ldots, X_d)'$ be an $d$-dimensional random vector and let $D \in \mathbb{R}^{d \times n}$ be a deterministic matrix with $d_{i1} < \cdots < d_{in}$ for all $i \in \{1, \ldots, d\}$.

Denote $d_{i0} = -\infty$ and $d_{i(n+1)} = \infty$.

Then we set

$$S_i = j \iff d_{ij} < X_i \leq d_{i(j+1)}$$
Suppose $X_i \sim F_i$ and $p_i = F_i(d_{i1})$ are its default probabilities.

By Slar’s theorem there exists a copula $C : [0, 1]^d \rightarrow [0, 1]$ such that, for all $x_1, \ldots, x_d$ in $\mathbb{R}^* = [-\infty, \infty]$, 

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$$

For any copula we get a joint distribution function.
CreditMetrics Model

- $X$ is assumed to have a multivariate normal distribution with standard normal margins, given by

$$X = BF + \varepsilon$$

- Here $F \sim N_p(0, \omega)$ with $p < d$, and $B \in \mathbb{R}^{d \times p}$ is a loading matrix.
- $\varepsilon_i$ are independent univariate normally distributed, independent of $F$.
- This construction automatically leads to the Gaussian copula.
Li’s Model

- Introduced in Li (2001)
- $X_i$, representing the default time of company $i$, is assumed to be exponentially distributed with parameter $\lambda_i$
- Then $F_i(t) = 1 - \exp(-\lambda_i t)$ and the $T$-years probability of default is $p_i = F_i(T)$
- In this model, $X$ is assumed to have the Gaussian copula $C_{d,\Gamma}$ for some variance matrix $\Gamma$
- Then

$$
\mathbb{P}(X_1 < t_1, \ldots, X_d < t_d) = C_{d,\Gamma}(F_1(t_1), \ldots, F_d(t_d))
$$
Which Copula to Use?

- In the case of the CreditMetrics model the Gaussian copula came out naturally.
- In the case of the Li’s model the appropriate copula was just a question of choice.
- The Gaussian copula in the Li’s model is often denoted as one of the drivers to the world financial crisis.
Contents

1 Duo Basket Example
2 Copula Functions
3 Portfolio Credit Risk
4 Tail Dependence and Limit Behavior
Tail Dependence

**Definition 4**

Let $X_1$ and $X_2$ be random variables with distribution functions $F_1$ and $F_2$. The coefficient of lower tail dependence of $X_1$ and $X_2$ is given by

$$
\lambda_l(X_1, X_2) = \lim_{q \to 0^+} P \left( X_2 \leq F_2^{-1}(q) | X_1 \leq F_1^{-1}(q) \right). \quad (26)
$$
Proposition 6

For continuous $F_1$ and $F_2$ there exist the unique copula $C$ such that

$$
\lambda_l(X_1, X_2) = \lim_{q \to 0^+} \frac{C(q, q)}{q}.
$$

(27)

Proof.

According to Sklar’s theorem

$$
\lambda_l(X_1, X_2) = \lim_{q \to 0^+} \mathbb{P} \left( X_2 \leq F_2^{-1}(q) | X_1 \leq F_1^{-1}(q) \right) =
$$

$$
= \lim_{q \to 0^+} \mathbb{P} \left( \frac{X_1 \leq F_1^{-1}(q), X_2 \leq F_2^{-1}(q)}{\mathbb{P} \left( X_1 \leq F_1^{-1}(q) \right)} \right) =
$$

$$
= \lim_{q \to 0^+} \frac{C(q, q)}{q}.
$$
Let $X_1, \ldots, X_d$ be random variables with common continuous (marginal) distribution function $F_1$ which admit an exchangeable copula. Then we can define

$$\lambda_1^1(X_1, \ldots, X_d) = \lim_{q \to 0^+} \mathbb{P}\left(X_1 \leq F_1^{-1}(q) | X_2 \leq F_1^{-1}(q), \ldots, X_d \leq F_1^{-1}(q)\right) =$$

$$= \lim_{q \to 0^+} \frac{C^d(q, \ldots, q)}{C^{d-1}(q, \ldots, q)}$$

$$\lambda_2^1(X_1, \ldots, X_d) = \lim_{q \to 0^+} \mathbb{P}\left(X_2 \leq F_1^{-1}(q), \ldots, X_d \leq F_1^{-1}(q) | X_1 \leq F_1^{-1}(q)\right) =$$

$$= \lim_{q \to 0^+} \frac{C^d(q, \ldots, q)}{q}$$
Multivariate Tail Dependence

Denote $A_i = \{X_i \leq F_1^{-1}(q)\}$

$$\lambda_3^3(X_1, \ldots, X_d) = \lim_{q \to 0^+} \mathbb{P}\left(\min(X_2, \ldots, X_d) \leq F_1^{-1}(q) | X_1 \leq F_1^{-1}(q)\right) =$$

$$= \lim_{q \to 0^+} \mathbb{P}\left(\bigcup_{i=2}^d A_i | A_1\right) =$$

$$= \lim_{q \to 0^+} \sum_{k=1}^{d-1} (-1)^{k+1} \sum_{2 \leq i_1 < \cdots < i_k \leq d} \mathbb{P}\left(\bigcap_{j=1}^k A_{i_j} | A_1\right) =$$

$$= \lim_{q \to 0^+} \frac{1}{q} \sum_{k=1}^{d-1} (-1)^{k+1} \binom{n-1}{k} C^{k+1}(q, \ldots, q)$$
We try to follow and generalize the approach of Lucas et al. (2003)

Assume for \( i \in \{1, \ldots, d\} \), \( \varepsilon_i \) iid, \( \theta_i \) iid, mutually independent, independent of \( f \)

Then \( X_i = g(f, \varepsilon_i) \) and default occurs when \( X_i < d_1 \)

In case of default we suffer a loss of \( Y_i = h(f, \varepsilon_i, \theta_i) \)
Average loss for $d$ obligors is given by

$$C_d = \frac{1}{d} \sum_{i=1}^{d} Y_i \cdot 1\{X_i < d_1\} \quad (28)$$

Does $C_d$ converge for $d \to \infty$? Under what assumptions?

What is the distribution of $C = \lim_{d \to \infty} C_d$? What are its tail properties?
Since around 1995 the copula approach to estimate portfolio credit risk has been rapidly growing, revealing its good and bad properties. Despite several alarming reports such as Embrechts et al. (2002), there was put too much trust into these simple models. One of the causes of the world financial crisis. A few possible directions for further studying:
- The choice of copula and model
- Proper dependence measures
- Limit behavior of large portfolios and losses
References


Thank you for your attention.