Generalized quantiles as risk measures

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Quantiles $q_\alpha$ of a random variable $X$ can be defined as the minimizers of a piecewise linear loss function:

$$q_\alpha(X) = \arg \min_{x \in \mathbb{R}} \{ \alpha \ E [(X - x)^+] + (1 - \alpha) \ E [(X - x)^-] \}.$$ 

This property lies at the heart of quantile regression (Koenker, 2005) and has been used by Rockafellar and Uryasev (2002) for the computation of the CVaR.
Generalized quantiles

have been introduced, by considering more general loss functions:

- expectiles (Newey, Powell; 1987)
- M-quantiles (Breckling, Chambers; 1988)
- $L^p$-quantiles (Chen; 1996)

General asymmetric loss function:

$$
\pi_\alpha(X, x) = \alpha \ E \left[ \Phi_1 \left( (X - x)^+ \right) \right] + (1 - \alpha) \ E \left[ \Phi_2 \left( (X - x)^- \right) \right],
$$

where $\Phi_1, \Phi_2$ are convex. Minimizer

$$
x^*_\alpha \in \arg \min_{x \in \mathbb{R}} \pi_\alpha(X, x)
$$

is called \textit{generalized quantile}. 
M-quantiles

Given a sample \( \{x_1, \ldots, x_n\} \) of univariate observations, with empirical distribution function \( F_n(x) \), the sample lower quartile \( \hat{q}_{1/4} \) is obtained as the solution of

\[
\int \psi_{1/4}(x - \hat{q}_{1/4}) F_n(dx) = 0,
\]

where

\[
\psi_{1/4}(x) = \begin{cases} 
\frac{3}{4} \text{sgn}(x) & (x < 0), \\
\frac{1}{4} \text{sgn}(x) & \text{otherwise}.
\end{cases}
\]

For arbitrary \( \alpha \) (0 < \( \alpha \) < 1) and any standard influence function \( \psi(x) \), is reasonable to define the influence function of the \( \alpha \)th M-quantile \( \theta_\alpha \) as follows:

\[
\psi_\alpha(x) = \begin{cases} 
(1 - \alpha)\psi(x) & (x < 0), \\
\alpha\psi(x) & \text{otherwise},
\end{cases}
\]

which leads to the estimating equation

\[
\int \psi_\alpha(x - \hat{\theta}_\alpha) F_n(dx) = 0.
\]
**$L^p$-quantiles**

Let $X$ be a random variable with cumulative distribution function $F(x)$. For $0 < \alpha < 1$, $1 \leq p < \infty$, define loss function

$$\rho^p_\alpha(x) = |\alpha - 1_{\{x < 0\}}||x|^p.$$

The $\alpha$th $L^p$-quantile of $X$ is defined as the minimizer $\mu_\alpha$ of

$$\int \rho^p_\alpha(x - \mu) dF(x).$$

The minimizer exists and is unique. Let

$$\psi^p_\alpha(x) = \begin{cases} (1 - \alpha)|x|^{p-1} & (x < 0), \\ -\alpha|x|^{p-1} & \text{otherwise}, \end{cases}$$

note that

$$\psi^p_\alpha(x - \mu) = \frac{1}{p} \frac{\partial}{\partial \mu} \rho^p_\alpha(x - \mu)$$

and that the minimizer is the solution of

$$\int \psi^p_\alpha(x - \mu) dF(x) = 0.$$
Elicitability

For evaluation of point forecasts, Gneiting (2011) introduced the notion of *elicitability* for a functional that is defined by means of loss minimization process. All generalized quantiles are elicitable. The relevance of elicitability for backtesting was addressed by Embrechts and Hofert (2013). It is well known how to backtest VaR, while backtesting of CVaR (that is not elicitable) is not as straightforward.

The connections between elicitability and coherence are also investigated in Ziegel (2013), who shows that expectiles are the only elicitable law-invariant coherent risk measures.
Law-invariant risk measures

Law invariance means that the risk assessment only depends on the distribution of the random variable under the given probability measure, without regard to the financial context.

Definition: A monetary risk measure $\rho$ on $L^\infty(\Omega; \mathcal{F}; P)$ is called law-invariant if $\rho(X)$ only depends on the distribution of $X$ under $P$, i.e. $\rho(X) = \rho(Y)$ whenever $X$ and $Y$ have the same distribution under $P$. 
Luxemburg norm

Given $\Phi : [0, +\infty) \to [0, +\infty)$ convex, strictly increasing function satisfying $\Phi(0) = 0$, $\Phi(1) = 1$, a probability space $(\Omega, \mathcal{F}, P)$ and the space $L^0$ of all r.v. $X$ on $(\Omega, \mathcal{F}, P)$ the Orlicz heart

$$M^\Phi := \left\{ X \in L^0 : E \left[ \Phi \left( \frac{|X|}{a} \right) \right] < +\infty, \text{ for every } a > 0 \right\}$$

is a Banach space w.r.t. the Luxemburg norm $\| \cdot \|_\Phi$, defined as

$$\| Y \|_\Phi := \inf \left\{ a > 0 : E \left[ \Phi \left( \frac{|X|}{a} \right) \right] \leq 1 \right\}.$$ 

The case $\Phi(x) = x^p$ corresponds to the usual $L^p$ spaces.
Properties of generalized quantiles

Since minimization problem of $\pi_\alpha(X, x)$ is convex, generalized quantiles can be characterized by means of first-order condition.

**Proposition 1.** Have $\pi_\alpha(X, x)$ and $\Phi_i$ as earlier. Let $X \in M^{\Phi_1} \cap M^{\Phi_2}$ and $\alpha \in (0, 1)$.

(a) $\pi_\alpha(X, x)$ is finite, non-negative, convex and satisfies
$$\lim_{x \to \pm \infty} \pi_\alpha(X, x) = +\infty;$$

(b) the set of minimizers is a closed interval:
$$\arg \min \pi_\alpha(X, x) := [x_{\alpha^-}^*, x_{\alpha^+}^*];$$

(c) $x_{\alpha^*}^\alpha \in \arg \min \pi_\alpha(X, x)$ iff (f.o.c.)
$$\alpha \ E \left[ \mathbb{I}_{\{X > x_{\alpha}^*\}} \Phi_i' (\delta^+) \right] \leq (1 - \alpha) \ E \left[ \mathbb{I}_{\{X \leq x_{\alpha}^*\}} \Phi_i' (\delta^-) \right]$$
$$\alpha \ E \left[ \mathbb{I}_{\{X \geq x_{\alpha}^*\}} \Phi_i' (\delta^+) \right] \geq (1 - \alpha) \ E \left[ \mathbb{I}_{\{X < x_{\alpha}^*\}} \Phi_i' (\delta^-) \right],$$

where $\Phi_i'$ and $\delta_X = X - x_{\alpha}^*$

(d) if $\Phi_1$ and $\Phi_2$ are strictly convex, then $x_{\alpha^-} = x_{\alpha^+}^*$
First order condition

**Example 2.** For $\Phi_1(x) = \Phi_2(x) = x$, generalized quantiles reduce to the usual quantiles and f.o.c. becomes

$$\alpha \ E \left[ I\{X > x_{\alpha}^*\} \right] \leq (1 - \alpha) \ E \left[ I\{X \leq x_{\alpha}^*\} \right]$$

$$\alpha \ E \left[ I\{X \geq x_{\alpha}^*\} \right] \geq (1 - \alpha) \ E \left[ I\{X < x_{\alpha}^*\} \right],$$

or, equivalently,

$$P(X < x_{\alpha}^*) \leq \alpha \leq P(X \leq x_{\alpha}^*).$$

**Corollary 3.** Under assumptions of Proposition 1, let $\Phi_i$ be differentiable. If $\Phi'_1(0) = \Phi'_2(0) = 0$ or the distribution of $X$ is continuous, the f.o.c. reduces to

$$\alpha \ E \left[ \Phi'_1 \left( (X - x_{\alpha}^*)^+ \right) \right] = (1 - \alpha) \ E \left[ \Phi'_2 \left( (X - x_{\alpha}^*)^- \right) \right].$$
Connection to shortfall risk measures

When the f.o.c. is given by an equation, generalized quantiles may also be defined as the unique solutions of the equation

\[ E [\psi(X - x^*_\alpha)] = 0, \]

where

\[ \psi(t) = \begin{cases} 
-(1 - \alpha)\Phi_2'(-t) & t < 0 \\
\alpha \Phi_1'(t) & t \geq 0 
\end{cases} \]

is nondecreasing with \( \psi(0) = 0. \)

This shows that generalized quantiles can be seen as special cases of zero utility premium principles, also known as shortfall risk measures or u-mean certainty equivalents (see Deprez, Gerber, 1985; Follmer, Schied, 2002; Ben-Tal, Teboulle, 2007).
Zero utility premium

Suppose that $u(.)$ is the insurer's utility function, with the usual properties (increasing, concave), and that $z$ represents the insurer's fortune without the new policy. Then the premium $P = H(X)$ is determined from the condition that

$$E[u(z + P - X)] = u(z),$$

which is the requirement that the premium should be fair in terms of utility. In the case of exponential utility,

$$u(x) = \frac{1}{a} (1 - e^{-ax}),$$

with parameter $a > 0$, equation has an explicit solution; one finds that

$$P = \frac{1}{a} \ln E(e^{aX}),$$

which is called the **exponential principle**.
One of certainty equivalents based on utility functions is the so-called \textit{u-mean}, $M_u(\cdot)$, defined for any random variable $X$ by $M_u(\cdot)$ satisfying

$$E[u(X - M_u(X))] = 0.$$ 

This equation is also known as the principle of zero utility.

As an example, the u-mean $M_u(\cdot)$ is closely related to the risk measure called \textit{shortfall risk} introduced by Follmer and Schied (2002), and defined by

$$\rho_{FS}(X) = \inf\{m \in \mathbb{R} : E[u(X - m)] \leq x_0\}.$$ 

For a strictly increasing utility and $x_0 = 0$, $\rho_{FS}(X) = M_u(X)$. 
Properties of generalized quantiles

**Proposition 6.** Let \( \Phi_i : [0, +\infty) \to [0, +\infty) \) be strictly convex and differentiable with \( \Phi_i(0) = \Phi_i'(0) = 0 \) and \( \Phi_i(1) = 1 \). Let \( \alpha \in (0, 1) \) and

\[
x_{\alpha}^*(X) = \arg\min_{x \in \mathbb{R}} \left\{ \alpha \ E \left[ \Phi_1 \left( (X - x)^+ \right) \right] + (1 - \alpha) \ E \left[ \Phi_2 \left( (X - x)^- \right) \right] \right\}.
\]

(a) \( x_{\alpha}^*(X) \) is positively homogeneous iff \( \Phi_i(x) = x^\beta \), with \( \beta > 1 \).

(b) \( x_{\alpha}^*(X) \) is convex (concave) iff the function \( \psi : \mathbb{R} \to \mathbb{R} \) is convex (concave).

(c) \( x_{\alpha}^*(X) \) is coherent iff \( \Phi_i(x) = x^2 \) and \( \alpha \geq \frac{1}{2} \).

Thus expectiles with \( \alpha \geq \frac{1}{2} \) are the only generalized quantiles that are coherent risk measure.
Expectiles $e_\alpha(X)$

The f.o.c. could be written in several equivalent ways:

$$\alpha \ E \left[(X - e_\alpha(X))^+\right] = (1 - \alpha) \ E \left[(X - e_\alpha(X))^-\right],$$

$$e_\alpha(X) - E[X] = \frac{2\alpha - 1}{1 - \alpha} \ E \left[(X - e_\alpha(X))^+\right],$$

$$\alpha = \frac{E \left[(X - e_\alpha(X))^-\right]}{E \left[|X - e_\alpha(X)|\right]}.$$

The latter shows, that expectiles can be seen as the usual quantiles of a transformed distribution (Jones, 1994).

Proposition 7. Let $X, Y \in L^1$, then:

(a) $X \leq Y$ $P$-a.s. and $P(X < Y) > 0$ imply that $e_\alpha(X) < e_\alpha(Y)$ (strong monotonicity);
(b) if $\alpha \leq \frac{1}{2}$, then $e_\alpha(X + Y) \geq e_\alpha(X) + e_\alpha(Y);$  
(c) $e_\alpha(X) = -e_{1-\alpha}(-X)$. 

Dual representation

as maximal expected value over a set of scenarios.

Proposition 8. Let $X \in L^1$, $\alpha \in (0, 1)$ and let $e_\alpha(X)$ be the $\alpha$-expectile of $X$. Then:

$$e_\alpha(X) = \begin{cases} 
\max_{\varphi \in \mathcal{M}_\alpha} E[\varphi X] & \alpha \geq \frac{1}{2} \\
\min_{\varphi \in \mathcal{M}_\alpha} E[\varphi X] & \alpha \leq \frac{1}{2},
\end{cases}$$

where

$$\mathcal{M}_\alpha = \left\{ \varphi \in L^\infty, \varphi \leq 0 \ a.s., E_P[\varphi] = 1, \frac{\text{ess sup } \varphi}{\text{ess inf } \varphi} \leq \beta \right\},$$

with $\beta = \max \left\{ \frac{\alpha}{1-\alpha}, \frac{1-\alpha}{\alpha} \right\}$. The optimal scenario is

$$\bar{\varphi} := \frac{\alpha I\{X > e_\alpha\} + (1 - \alpha)I\{X \leq e_\alpha\}}{E \left[ \alpha I\{X > e_\alpha\} + (1 - \alpha)I\{X \leq e_\alpha\} \right]}.$$
Kusuoka representation

From the dual representation it is possible to derive Kusuoka (2001) representation, which is the representation of law invariant coherent risk measure as a supremum of convex combinations of CVaR.

**Proposition 9.** Let $X \in L^1$, $\alpha \in \left[ \frac{1}{2}, 1 \right)$ and $\beta = \frac{\alpha}{1 - \alpha}$, then

$$e_\alpha(X) = \max_{\gamma \in \left[ \frac{1}{\beta}, 1 \right]} \left\{ (1 - \gamma) \text{CVaR}_{\frac{\beta \gamma - 1}{\gamma(\beta - 1)}} + \gamma E[X] \right\}.$$

In particular,

$$e_\alpha(X) \geq \frac{E[X]}{2\alpha} + \left( 1 - \frac{1}{2\alpha} \right) \text{CVaR}_\alpha(X).$$
Robustness

In robust statistics, the notion of qualitative robustness of a statistical functional corresponds essentially to the continuity with respect to weak convergence. Coherent risk measures are not robust in statistical sense. Stahl et al. (2012) suggest that a better notion of robustness might be continuity with respect to the Wasserstein distance, defined as

\[ d_W(P, Q) := \inf \{ E[|X - Y|] : X \sim P, Y \sim Q \} \,.
\]

Convergence in the Wasserstein distance is stronger than weak convergence:

\[ d_W(X_n, X) \to 0 \iff X_n \to X \text{ in distribution and } E[X_n] \to E[X]. \]

**Theorem 10.** For all \( X, Y \in L^1 \) and all \( \alpha \in (0, 1) \) holds that

\[ |e_\alpha(X) - e_\alpha(Y)| \leq \beta \ d_W(X, Y), \text{ where } \beta = \max \left\{ \frac{\alpha}{1-\alpha}, \frac{1-\alpha}{\alpha} \right\}. \]
Comparing expectiles with quantiles

Koenker (1993) provided an example of a distribution with infinite variance for which expectiles $e_\alpha(X)$ and quantiles $q_\alpha(X)$ coincide for all $\alpha \in (0, 1)$. This distribution is Pareto-like with tail index $\beta = 2$.

**Theorem 11.** Assume that $X$ has a Pareto-like distribution with tail index $\beta > 1$. Then

$$\frac{\bar{F}(e_\alpha(X))}{\beta - 1} \sim 1 - \alpha \sim \bar{F}(q_\alpha(X)) \quad \text{as } \alpha \to 1.$$ 

If $\beta < 2$, then there exists some $\alpha_0 < 1$ such that for all $\alpha > \alpha_0$ $e_\alpha(X) > q_\alpha(X)$ holds; if $\beta > 2$, the reverse inequality applies. So for high $\alpha$ expectiles are more conservative than the quantiles for distributions with heavy tails (infinite variance).


