Stochastic Finance - A Numeraire Approach

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Stochastické modelování v ekonomii a financích

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Motivation:

- What are the ideas behind derivatives valuation?
- Modern Martingale (risk-neutral) approach to valuation may seem to appear from nowhere. The more unmotivated stays the Numeraire approach, which generalizes valuation so that the original motivation may stay blurred forever.
- However, the Numeraire approach turns out to give very elegant results which seem to appear easily.

Outline:

1. Go through the most standard discrete-time models, show why we should accept that prices are some special expectations and that prices should be modeled as martingales for derivative pricing purposes.
2. Introduce the concept of Numeraire, revisit the Binomial model and prepare ground for lectures in continuous time.
**Definition**

*Price* \( X_Y(t) \) at time \( t \) is a number representing how many units of an asset \( Y \) are required to obtain one unit of an asset \( X \). The reference asset \( Y \) is called a *numeraire*. Price is always a pairwise relationship.

Most often, *banknotes*, say \$, are chosen as a numeraire. An asset \( X \), say one Coca–Cola stock, has a price \( X_\$ \cdot \$. However, not only easily tradeable assets are chosen as numeraires—\( \max_{0 \leq s \leq t} X_Y(s) \) can again be a numeraire.

Assets do not have numerical value on their own. We may write \( X(t) = Z(t) \) in the sense of assets (= is rather to be read as an equivalence) when \( X_Z(t) = 1 \) in the sense of numbers.
Once we have a price of an asset $X_Y(t)$, we have even an inverse price $Y_X(t)$:

$$X(t) = X_Y(t) \cdot Y(t) = X_Y(t)Y_X(t) \cdot X(t) \quad \Rightarrow \quad Y_X(t) = X_Y(t)^{-1}$$

Think of exchange rates as a good motivation. One dollar note does not change in time, once we express its price in terms of one euro note, this number may change continuously.

Another simple observation:

$$X(t) = X_Y(t) \cdot Y(t)$$
$$X(t) = X_Z(t) \cdot Z(t) = X_Z(t)Z_Y(t) \cdot Y(t) \quad \Rightarrow \quad X_Y(t) = X_Z(t)Z_Y(t)$$

which we call a “change of numeraire formula”
Motivation for Numeraire Approach

Introduction to Valuation via Numeraires

Definition

Let $\Delta^i(t)$, a predictable process, represent a number of $X^i(t)$ units held at time $t$. A portfolio and a portfolio price are given by:

$$P(t) = \sum_{i=1}^{n} \Delta^i(t)X^i(t), \quad P_Y(t) = \sum_{i=1}^{n} \Delta^i(t)X^i_Y(t).$$

Definition

Let $P(0) = 0$ and follow a self-financing strategy (at any time one can exchange only equivalent assets).

If there exists an arbitrary $T > 0$, such that $P(T) \geq 0$ a.s. and $P(P(T) > 0) > 0$ (i.e. with positive probability the portfolio is equivalent to an asset which is not worthless), we say that an arbitrage exists in a market.
If an arbitrage is not possible by holding a certain asset, we call the asset a *no–arbitrage asset*.

**Observation**

Let $V$ be a contract to deliver $X$ at time $T$. Obviously $V(T) = X(T)$. For a no–arbitrage asset, it holds that $V(t) = X(t) \forall t \in [0, T]$.

**Observation**

The contract to deliver an asset $X$ is always a no–arbitrage asset, no matter whether $X$ is a no–arbitrage asset or not.

Further, unless stated otherwise, a portfolio consists of no–arbitrage assets.
Definition

A *Contingent Claim* is any random variable $C$ that is defined on a stochastic basis and $E|C| < \infty$.

Definition

A *Derivative (Security)* of an asset $S$ is a contingent claim such that it is $\sigma_\infty(S)$–measurable.

Definition

A (European Vanilla) *Call Option* on $S$ with strike $K$ and maturity $T$ is a derivative such that at time $T$

$C_\$ (T, S(T)) = (S_\$ (T) – K)^+$
Self–financing portfolio in discrete time: Price transition happens just before the portfolio is rebalanced so the following must hold:

\[ P_Y(t + 1) - P_Y(t) = \sum_{i=0}^{N} \Delta^i(t) [X_Y^i(t + 1) - X_Y^i(t)] \]

\[ \sum_{i=0}^{N} [\Delta^i(t + 1) - \Delta^i(t)] X_Y^i(t + 1) = 0 \]

Self–financing portfolio in continuous time: We postulate:

\[ dP_Y(t) = \sum_{i=0}^{N} \Delta^i(t) dX_Y^i(t) \]

By Ito formula (note the similarity to the discrete case), it must also hold that

\[ \sum_{i=0}^{N} [d[\Delta^i, X_Y^i](t) + d\Delta^i(t)X_Y^i(t)] = 0. \]
Examples of Self–financing portfolios

\[ P(t) = \left[ 1 - \frac{t}{T} \right] X + \left[ \frac{1}{T} \int_0^t X_Y(s)ds \right] Y \]

is a self–financing portfolio.

\[ P(t) = \Phi(d_+)X + [-K\Phi(d_-)]Y, \]

where

\[ d_\pm = \frac{1}{\sigma \sqrt{T - t}} \log \left( \frac{X_Y(t)}{K} \right) \pm \frac{1}{2} \sigma \sqrt{T - t}, \]

is a self–financing portfolio too. (It is a portfolio replicating a European vanilla call option)
The following four lectures concentrate on the valuation of derivatives – contracts such as a vanilla option.

Imagine we live in a simple world with one bond and one stock. There are only two time steps $0$ and $T$. The bond evolves as $B_\$ (T) = (1 + r)B_\$(0)$ and the stock evolves randomly – on a one period binomial lattice:

- $S_\$ (T) = uS_\$(0)$ with probability $p$;
- $S_\$ (T) = dS_\$(0)$ with probability $q = 1 - p$.

How would we price a vanilla option, which pays $(S_\$(T) - K)^+$?

40 years ago, we would have computed an expectation. Black and Scholes have shown such an approach is wrong!

First note, that to exclude arbitrage, it must hold that $d < 1 + r < u$. ($1 + r < d < u$: Sell a Bond, get $B_\$(0)$ banknotes, buy a fraction of stock and realize a risk free profit.)
Black and Scholes (with Merton adding mathematical rigour) formalized the idea of replication. This idea in discrete one-period model (by Cox, Ross and Rubinstein) looks as follows: Without loss of generality assume $B(0) = 1$. We would like a portfolio consisting of $S$ and $B$ be such that it replicates the option pay–off, i.e. it is true that

$$V_S(\Delta, T) = \Delta^S(0)S_S(T) + \Delta^B(0)B_S(T) = C_S(T, S(T))$$

i.e.

$$\Delta^S(0)uS_S(0) + \Delta^B(0)B_S(T) = C_S(T, uS(0))$$
$$\Delta^S(0)dS_S(0) + \Delta^B(0)B_S(T) = C_S(T, dS(0))$$

If such a portfolio exists, than by no–arbitrage arguments the dollar price of the option at time 0 must equal the dollar price of the portfolio at time 0—$\Delta^S(0)S_S(0) + \Delta^B(0)$
Watch out! We found a price without a word about the probability of the Stock moving up. Actually we didn’t need the word probability at all. We needed replication only. Our two equations of two unknowns ($\Delta^S$ and $\Delta^B$) imply

$$\Delta^S(0)S_0(u - d) = C_0(T, uS_0) - C_0(T, dS_0)$$
$$\Delta^B(0)B_0(T)(d - u) = C_0(T, uS_0)d - C_0(T, dS_0)u$$

which in turn implies

$$\Delta^S(0) = \frac{C_0(T, uS_0) - C_0(T, dS_0)}{S_0(u - d)};$$
$$\Delta^B(0) = \frac{C_0(T, uS_0)d - C_0(T, dS_0)u}{B_0(T)(d - u)}$$
However, to have the theory elegant, there should be a way to compute prices as expectations! For this we introduce an auxiliary measure $p^*$, which makes the bond–discounted stock dollar price process a martingale. The discounting procedure makes a good economic sense and in the general setting must be done to avoid mispricing due to banknotes being an arbitrage asset. Also, we will see that it means the stock bond–price process is then a martingale (via change of numeraire). By the change of numeraire we also see, that the martingale measure really depends on the specific numeraire.

\[
S_\$ (T) = \frac{1}{1 + r} (p^* u S_\$(0) + (1 - p^*) d S_\$(0))
\]

hence $p^* = \frac{1 + r - d}{u - d}$, $1 - p^* = \frac{u - (1 + r)}{u - d}$. Note that this is a probability measure only because of our assumptions on $u, d, r$. If these hadn’t hold, we would have not been able to find a probability measure! (no arbitrage $\Rightarrow \exists$ martingale measure) is true in general, see the First Fundamental Theorem.
Now, from the replication principle, we remember, that
\[
V_S(0) = \Delta^S(0) S_S(0) + \Delta^B(0)
\]
\[
= \frac{C_S(T, uS_S(0)) - C_S(T, dS_S(0))}{S_S(0)(u - d)} S_S(0)
\]
\[
+ \frac{C_S(T, uS_S(0)) d - C_S(T, dS_S(0)) u}{(1 + r)(d - u)}
\]
\[
= \frac{1}{1 + r} \left( \frac{1 + r - d}{u - d} C_S(T, uS_S(0)) + \frac{u - (1 + r)}{u - d} C_S(T, dS_S(0)) \right).
\]
which in the language of our "artificial" martingale measure reads as
\[
V_S(0) = \frac{1}{1 + r} (p^* C(T, uS_S(0)) + (1 - p^*) C_S(T, dS_S(0)))
\]
\[
= E^* \left( \frac{C_S(T, S_S(T))}{1 + r} \right).
\]
Price of the option can be expressed as an expectation with respect to a measure given by the numeraire. (∃∃)
Motivation for Numeraire Approach

Introduction to Valuation via Numeraires

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Numeraire Valuation
Let us now move to pricing via numeraires nad martingales in general. Hopefully, now it is clear why it makes sense to do it.
Let us now move to pricing via numeraires nad martingales in general. Hopefully, now it is clear why it makes sense to do it. When modeling a financial market, we do not want arbitrage to exist in the model—simply because in reality arbitrage due to economic reasons (almost) doesn’t exist. If it appears (a contract is mispriced), it is immediately exploited and the prices are moved so that an arbitrage is not possible anymore.

Fortunately, there is a mathematical rule, which says whether our model would allow arbitrage or not. It is called First Fundamental Theorem of Asset Pricing.

It says, that whatever our probabilistic model of financial markets (specifically assets in them) may look like, for it to be reasonable and not allow arbitrage means the same as that there must exist a probability measure such that it makes all processes of asset prices with respect to a chosen numeraire martingales!
First Fundamental Theorem of Asset Pricing

**Theorem**

∃ \( P^Y \) a probability measure s.t. \( X^Y_i(t) \) are \( P^Y \)-martingales for \( X \) an arbitrary no-arbitrage asset and \( Y \) a no-arb. asset which is never worthless ⇐⇒ ∄ arbitrage in the market.

**Proof.**

Prove the converse of one implication only: If there were an arbitrage, \( P^Y(t) \) wouldn’t be a \( P^Y \)-martingale. \( P^Y(T) = \Psi \), where \( \Psi \) is a non-negative r.v. s.t. \( P^Y(\Psi > 0) > 0 \)

This implication is enough for practical purposes. When we model price evolutions, we simply check that we are able to make them martingales. (We may for example only adjust parameters for such a condition to hold).
Interpretation of the martingale probability measure $P_Y(t)$: Let $V$ be an *Arrow–Debreu* security, i.e. a contract that pays one unit of numeraire when a certain state is reached:

$$V(T) = \mathbb{I}_A(\cdot) Y(T), \quad \text{i.e.} \quad V_Y(T) = \mathbb{I}_A(\cdot)$$

$V_Y(t)$ is a $P_Y$–martingale, therefore

$$V_Y(0) = E^Y[V_Y(T)] = P_Y(A).$$

Clearly, the initial market price of the Arrow–Debreu security lies in $[0,1]$ and the martingale probability indicates how “costly” is an event $A$ rather than how likely. The price corresponding to the Arrow–Debreu security for a single scenario in discrete state space is known as an *Arrow–Debreu* state price. In continuous time, we generalize this concept to

$$P_Y(A) = \int_A dP^Y$$

and to the so–called *state price density*. 
Pricing basis

Pricing of basic no–arbitrage assets is easy. The only way to deliver them is to hold them, hence a contract delivering on unit of assets $X$ at time $T$ must be worth $X(t)$ at any prior time. (Otherwise there would be an arbitrage) Imagine a Money Market Account or a Bond being a numeraire and note that banknotes are arbitrage assets!

Further we try to exploit the idea of prices of no–arbitrage assets for pricing more complicated contracts. We try to price contracts of the form $f(X(T), V(T))$ for two no–arbitrage assets $X$ and $Y$.

Once again–we saw that an arbitrage is not possible when there is a martingale measure. We also saw, that in such a case, a price of such a contract can be computed as an expectation w.r.t. the martingale measure. But what if there are more martingale measures?
One–period Trinomial model

The price $X_Y(T)$ is assumed to take three possible values: $uX_Y(0)$, $X_Y(0)$ or $dX_Y(0)$, corresponding to scenarios. With suitably chosen parameters (which?) the following measure

$$P^Y(uX_Y(0)) = \frac{1-d}{u-d} \xi, \quad P^Y(X_Y(0)) = 1-\xi, \quad P^Y(uX_Y(0)) = \frac{u-1}{u-d} \xi$$

makes the price process $X_Y(t)$ a martingale. Therefore, there are actually uncountably many martingale measures! Consider an Arrow–Debreu security, which pays one unit of $Y$ if $X_Y(T) = uX_Y(0)$. If we relied on the expectation w.r.t. the martingale measure, we get that the initial price is equal to $P^Y(uX_Y(0))$. This is not feasible, we don’t want uncountably many prices to be the correct ones!
Observation

*Binomial model is a special case of $\xi = 1$. The martingale measure is thus unique in the binomial model.*

When we get back to our original “economic” replication approach of a derivative $V_Y(X, Y)$, how does it apply? It leads generally to three linear equations with to unknowns:

\[
V_Y(T, uX) = \Delta^X(0)uX_Y(0) + \Delta^Y(0)
\]
\[
V_Y(T, X) = \Delta^X(0)X_Y(0) + \Delta^Y(0)
\]
\[
V_Y(T, dX) = \Delta^X(0)dX_Y(0) + \Delta^Y(0).
\]

This means we are not able to replicate the derivative.

Definition

A market model, where every derivative can be replicated, is called *complete*. It is called *incomplete* otherwise.
Assume however, that an investor believes a trinomial market is a good model. What do they do to find a price for a derivative? One possibility is to introduce another underlying asset and complete the model. For example, assume the market trades an Arrow–Debreu security $Z$, that pays one unit of $Y$ if $X_Y(1) = uX_Y(0)$ happens. Remember, the quote of $Z_Y(0)$ tells us which martingale measure is the right one. The market is then complete if we consider a portfolio

$$P(0) = \Delta^X(0)X + \Delta^Y(0)Y + \Delta^Z(0)Z.$$ 

Such a portfolio yields three equations of three unknowns.
There may be more martingale measures in the model. As said, this unfortunately means, that despite the fact that there is no arbitrage, there might be more “correct” prices of a security. We would like to avoid such a situation. We know, that we can “manually” check, whether a market is complete simply by checking the replication equations for each individual derivative. Mathematically speaking, it seems like we cannot say upfront, whether our model is complete or not. Again, there is a way out of this quagmire:

**Theorem**

A market is complete if and only if the martingale measure $\mathbb{P}^Y$ is unique.

Proof is quite technical. In the continuous case, it is actually quite difficult, one must redefine arbitrage and plunge into functional analysis with $w^* -$ topologies etc.
To be continued...
Fundamental Theorems of Asset Pricing

Theorem (First Fundamental Theorem)

\[ \exists \mathbb{P}^{Y} \text{ a probability measure s.t. } X^{Y}_{t} \text{ are } \mathbb{P}^{Y} \text{-martingales for } X \text{ an arbitrary no-arbitrage asset and } Y \text{ a no-arb. asset which is never worthless } \iff \nexists \text{ arbitrage in the market.} \]

There may be more martingale measures in the model. As said, this unfortunately means, that despite the fact that there is no arbitrage, there might be more “correct” prices of a security. We would like to avoid such a situation. We know, that we can “manually” check, whether a market is complete simply by checking the replication equations for each individual derivative.
Mathematically speaking, it seems like we cannot say upfront, whether our model is complete or not. Again, there is a way out of this quagmire:

**Theorem (Second Fundamental Theorem)**

* A market is complete if and only if the martingale measure $P^Y$ is unique.
Change of measure

In an ideal world where a martingale measure is unique, we have by the change of numeraire formula:

\[ V = E^Y [V_Y(T)] \cdot Y(0) = E^X [V_X(T)] \cdot X(0) \]

Remember the Radon–Nikodym derivative: Martingale measures are equivalent, hence:

\[ E^X [V_X(T)] \cdot X(0) = E^Y [V_X(T)Z(T)] \cdot X(0) \]

Since this is true for an arbitrary payoff, we have a financial interpretation:

\[ V_X(T)Z(T) \cdot X(0) \Rightarrow Z(T) = \frac{V_Y(T)}{V_X(T)} Y_X(0) = \frac{X_Y(T)}{X_Y(0)} \]
Binomial model revisited

Ideal world with two time steps 0, T, two no–arbitrage assets X and Y and two possible outcomes for $X_Y(T) = uX_Y(0)$ or $uX_Y(0)$. We get $p^Y = \frac{1-d}{u-d}$. On the contrary, $Y_X(0)$ turns into $\frac{Y_X(0)}{u}$ or $\frac{Y_X(0)}{d}$, which yields $p^X = u\frac{1-d}{u-d}$. Of course, this result is easily deduced from the “financial interpretation” of Radon–Nikodym derivative.

Binomial model is easily modified to more periods.
Computing dollar prices: \( V_{BT}(0) = E^{BT} [V_B^T(T)] \), therefore, assuming \( B^T_\$(T) = 1 \) (this is more common and actually the only correct in full generality)

\[
V_\$(0) = V_{BT}(0)B^T_\$(0)
\]

\[
= E^{BT} [V_\$(T)B^T_\$(T)]B^T_\$(0) = E^{BT} [V_\$(T)]B^T_\$(0)
\]

This formula is of fundamental importance. If we find a corresponding martingale measure for prices evaluation w.r.t. a bond maturing at time \( T \), we can immediately price a contract without a need to find an interest rate, for example! We get the Bond dollar value directly from the market. This is not the case when we take a money market as a numeraire. Than we need to know the interest rate and it must be deterministic to be able to compute anything.
Exotic Assets

Even self–financing portfolios composed of no–arbitrage assets are (with some restrictions on $\Delta s,$) no–arbitrage assets and therefore come with their own martingale probability measure. Let us illustrate it on an asset:

$$A(T) = \frac{1}{2} [X_Y(0) + X_Y(T)]$$

Clearly, $\Delta^X(0) = \frac{1}{2}$ is the self–financing portfolio replicating the asset.

$$P^A = Z(T)P^Y = \frac{A_Y(T)}{A_Y(0)P^Y} = \frac{1}{2} \left( 1 + \frac{X_Y(T)}{X_Y(0)} P^Y \right)$$

This is a measure important in pricing all Asian options. Such an averaging portfolio is made a numeraire and computations turn out to be much simpler.
Motivation for Numeraire Approach
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Binomial model with an arbitrage asset

To be able to use the fundamental theorems, we must hedge only through no–arbitrage assets, otherwise of–course we have arbitrage whatever we might try to do. When the underlying assets are stock and dollar, we must hedge trought stock and bond, for example.

With American Options, this is generally not feasible, as they may be exercised at any time, hence we cannot choose bond maturities. Hence hedging in a bond provides us with a proxy only!

Consider $S_\$ $ and $B_\$ $ and their common evolution

$S_\$(T) = xS_\$(T − 1)$, where $x = u, d$, and

$B_\$(T) = (1 + r)B_\$(T − 1)$. We can express these evolutions in terms of a no arbitrage assets, for example the bond (but stock works too!) and get the familiar formulas.
American Options

American options can be exercised at any time. At any time, there is an *intrinsic* value of the option, the value of immediate exercise, and *continuation* value, the value of, well, continuation.

\[ V_S(T - 1) = \max \left( f_S(S_S(T)), \frac{1}{1 + r} \mathbb{E}_T^{B_T} [V_S(T + 1)] \right) \]

Informally, we “delete” some of the lower nodes of the tree. American options are much more elegant (mathematically speaking) in continuous time, where the theory of optimal stopping is much more interesting. (It isn’t only a backward induction).

This is it from the introduction. Let us see the continuous case.
Thank you for attention!

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