Weak Solutions to the Navier–Stokes–Fourier System on Lipschitz Domains

Lukáš Poul
poul@karlin.mff.cuni.cz

Mathematical Institute, Academy of Sciences of the Czech Republic, Prague
Faculty of Mathematics and Physics, Charles University, Prague

Abstract

We study existence of weak solutions to the Navier–Stokes–Fourier system on bounded Lipschitz domains in $\mathbb{R}^3$. For bounded smooth domains of class at least $C^{2+\nu}, \nu > 0$ there is an affirmative answer already due to the theory developed by Feireisl and others. The approach to the existence of solutions for merely Lipschitz domains is based on inserting additional limit passage into the existence proof that is already known (cf. [2], [3], for example).

Introduction

The immediate state of a viscous, compressible, and heat conducting fluid can be described by a triple of functions $(\rho, \mathbf{u}, \vartheta)$ representing physical quantities density, velocity, and temperature. Time-evolution of the system can be caught up by a system of partial differential equations representing the basic physical principles. They are: The continuity equation expressing the total mass balance of the system

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0. \tag{1}$$

The second Newton's law is written in the form of the linear momentum equation

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \operatorname{div}\mathbb{S} + \rho\mathbf{f}, \tag{2}$$

where p denotes the pressure and $\mathbb{S}$ denotes the Cauchy stress tensor. The exact form of p and $\mathbb{S}$ are given by constitutive relations. External forces are given by the term $\mathbf{f}$.
The first law of thermodynamics specifies the internal energy e as a conserved quantity, this is in fact equivalent with the entropy production equation.

$$\partial_t(\rho s) + \operatorname{div}(\rho s\mathbf{u}) + \operatorname{div}\frac{\mathbf{q}}{\vartheta} = \frac{\Phi}{\vartheta} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}, \tag{3}$$

where $\mathbf{q}$ denotes the heat flux and Φ expresses the dissipation of the mechanical energy into heat.
The constitutive relations describing quantities p, $\mathbb{S}$ and $\mathbf{q}$ are given as follows

$$\begin{aligned} p &= p(\rho,\vartheta) = p_e(\rho) + \vartheta p_\vartheta(\rho) + \frac{d}{3}\vartheta^4 \\ \mathbb{S} &= \mu(\vartheta)\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) + \lambda(\vartheta)(\operatorname{div}\mathbf{u})\mathbb{I} \\ &= \mu(\vartheta)\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \frac{2}{3}(\operatorname{div}\mathbf{u})\mathbb{I}\right) + \zeta(\vartheta)(\operatorname{div}\mathbf{u})\mathbb{I} \\ \mathbf{q} &= -\kappa(\vartheta)\nabla\vartheta \end{aligned}$$

If the motion is smooth, the linear momentum equation (2) can by multiplied by $\mathbf{u}$ in order to obtain $\Phi = \mathbb{S} : \nabla\mathbf{u}$. However, for nonsmooth motions only one inequality holds

$$\Phi \geq \mathbb{S} : \operatorname{div}\mathbf{u}. \tag{4}$$

The quantities p, s, and e are interrelated by Gibb's equation $Ds = De + pD(\frac{1}{\rho})$, where D represents the total derivative with respect to variables ρ and ϑ. Consequently, assuming moreover that the *specific heat at constant volume* c_v is constant, e and s have the form

$$e(\rho,\vartheta) = P_e(\rho) + d\frac{\vartheta^4}{\rho} + c_v\vartheta,$$

$$s(\rho,\vartheta) = \frac{4}{3}d\frac{\vartheta^3}{\rho} + c_v\log\vartheta - P_\vartheta(\rho)$$

where $P_e(z) = \int_1^z \frac{p_e(s)}{s^2}\,\mathrm{d}s$ and $P_\vartheta(z) = \int_1^z \frac{p_\vartheta(s)}{s^2}\,\mathrm{d}s$.
We assume that there is no slip at the boundary and the system is thermally isolated, i.e.

$$\mathbf{u}|_{\partial\Omega} = 0, \text{ and } (\nabla\vartheta\cdot\mathbf{n})|_\Omega = 0.$$

Structural Assumptions

Pressure

$$p_e(0) = 0, \quad p_e'(\rho) \geq a_1\rho^{\gamma-1} - c_1, \quad p_e(\rho) \leq a_2\rho^\gamma + c_2$$

$$p_\vartheta(0) = 0, \quad p_\vartheta'(\rho) \geq 0, \quad p_\vartheta(\rho) \leq a_3\rho^\Gamma + c_3,$$

where $a_1 > 0, \gamma \geq 2, \gamma > \frac{4\Gamma}{3}$.

Viscosity Terms

$$0 < \underline{\mu}(1+\vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta)^\alpha$$

$$0 < \underline{\zeta}\vartheta^\alpha \leq \zeta(\vartheta) \leq \overline{\zeta}(1+\vartheta)^\alpha$$

with $\frac{1}{2} \leq \alpha \leq 1$.

Heat Conductivity Term

$$0 < \underline{\kappa_G} \leq \kappa_G(\vartheta) \leq \overline{\kappa_G}(1+\vartheta^3), \quad \kappa_R(\vartheta) = \sigma\vartheta^3$$

Weak Solutions

The triple $(\rho, \mathbf{u}, \vartheta)$ is a solution of the Navier–Stokes–Fourier System if following holds

Renormalized Continuity Equation

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3)$$

for any $b \in C^1[0,\infty)$, $b'(z) = 0, z \geq z_0$, where z_0 depends on b.

Linear Momentum Equation

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho,\vartheta) = \operatorname{div}\mathbb{S} - \rho\mathbf{f} \text{ in } \mathcal{D}'((0,T)\times\Omega)$$

Entropy Production Inequality

$$\begin{aligned} &\partial_t(\rho s(\rho,\vartheta)) + \operatorname{div}(\rho s(\rho,\vartheta)\mathbf{u}) - \operatorname{div}\frac{\kappa(\vartheta)\nabla\vartheta}{\vartheta} \\ &\qquad \geq \frac{1}{\vartheta}\left(\mathbb{S}:\nabla\mathbf{u} + \frac{\kappa(\vartheta)|\nabla\vartheta|^2}{\vartheta}\right) \text{ in } \mathcal{D}'([0,T)\times\overline{\Omega}) \end{aligned}$$

Total Energy Equality

$$\int_0^T \partial_t\xi\int_\Omega\left(\frac{1}{2}\rho|\mathbf{u}|^2(t) + \rho e(\rho,\vartheta)(t)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t = -\int_0^T\int_\Omega \xi\rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}$$

for any $\xi \in C^\infty([0,T]), \xi(0) = 1, \xi(T) = 0$.

Main Ideas

(i) Mollify the system introducing artificial viscosity term $\varepsilon\Delta\rho$ in the continuity equation and artificial pressure term $\delta\rho^\beta, \beta \gg 1$ in the linear momentum equation (at this point the smoothness of the boundary is necessary for the parabolic estimates to hold).

(ii) Solve the system on a smooth domain for $\varepsilon, \delta > 0$ and pass with ε to zero.

(iii) Include the passage of Ω_n to Ω and obtain solution for system with the artificial pressure term on a bounded domain with merely Lipschitz continuous boundary.

(iv) Pass with δ to zero.

Main Result

Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz continuous boundary. Moreover, assume that the assumptions on the terms $p_e, p_\vartheta, \kappa, \lambda, \mu$ hold, and let $\mathbf{f} \in L^\infty((0,T)\times\Omega)$. Then for any initial conditions $\rho(0) = \rho_0 \geq 0$, $\rho_0 \in L^\gamma(\Omega)$, $(\rho\mathbf{u})(0) = \mathbf{m}_0 \in L^1(\Omega;\mathbb{R}^3)$, $\frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega)$, $\vartheta(0) = \vartheta_0 \in L^\infty(\Omega), \frac{1}{\vartheta_0} \in L^\infty(\Omega)$. there exists a weak solution to the Navier–Stokes–Fourier system on Ω.
Moreover, there exists a weak solution $(\rho, \mathbf{u}, \vartheta)$ enjoying the following properties: $\mathbf{u} \in L^r(0,T;W_0^{1,r}(\Omega)^3)$ for some $r > 1$; $\vartheta, \log\vartheta \in L^2(0,T;W^{1,2}(\Omega))$; $\rho \in C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^\gamma(\Omega))$; $\rho\mathbf{u} \in C([0,T];L_{weak}^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3))$; the quantities $\rho\mathbf{u}\otimes\mathbf{u}$, $\mathbb{S}:\nabla\mathbf{u}$, p, $\rho\mathbf{f}$ are integrable on $(0,T)\times\Omega$;

Remarks on Sensitivity with Respect to the Domain

- in the case when Ω is approximated by sequence of domains Ω_n, where $\Omega \subset \Omega_n$, the only problem is the question whether if $u_n \in W_0^{1,2}(\Omega_n)$ and $u_n \rightharpoonup u$ in $W^{1,2}(\mathbb{R}^N)$, then $u \in W_0^{1,2}(\Omega)$. However, this is true if Ω has Lipschitz continuous boundary.

- in the case when Ω is approximated by a sequence of domains Ω_n, where $\Omega_n \subset \Omega$, the problem with the boundary condition on the heat flow occurs. The limit passage (up to the autor's best contemporary knowledge) does not guarantee in general that the limit functions satisfy the entropy production inequality. However, if the approximation is taken such that

$$\operatorname{cap}_2(\Omega\setminus\Omega_n) \to 0 \text{ as } n \to \infty,$$

where $\operatorname{cap}_2(M) := \inf\{\int_{\mathbb{R}^3}|\nabla\varphi|^2\,\mathrm{d}\mathbf{x} : \varphi \in \mathcal{D}(\mathbb{R}^3), \varphi|_M \geq 1\}$ stands for the exterior capacity of the set M, then the entropy production inequality holds for the limit functions. This covers for example the case when Ω is Lipschitz and piecewise smooth (a cube).

Further Outlook – Existence in (more) general domains

The major problem in proving existence of solutions for spatial domains which boundary is not Lipschitz continuous is the lack of estimates on the so called Bogovskii operator (branch of solutions to $\operatorname{div}\mathbf{u} = g, \mathbf{u} \in W_0^{1,p}(\Omega;\mathbb{R}^3)$). This might be successively treated in the case of exterior domains for example, but the general case with boundary not necessarily compact stays open.

References

[1] Poul, L.: Weak Solutions to the Navier–Stokes–Fourier System on Bounded Lipschitz Domains. In preparation for Proceedings of the 6th AIMS Conference, Poitiers 2006.

[2] Feireisl, E. (2004): Dynamics of Viscous Compressible Fluids, *Oxford Lecture Series in Mathematics and Its Applications* **26**, Oxford University Press.

[3] Ducomet, B., Feireisl, E.: On the Dynamics of Gaseous Stars, *Arch. Rational Mech. Anal.* **174**, 221–266 (2004).

Acknowledgement: Work on this project was partially supported by Grant no. 6/2005/R of the Grant Agency of the Charles University, Prague, and by Grant no. A1019302 of the Grant Agency of the Czech Republic.