Weak Solutions to the Navier–Stokes–Fourier System on Lipschitz Domains

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Entropy Equation

$$\partial_t(\rho s) + \operatorname{div}(\rho s \mathbf{u}) - \operatorname{div} \frac{\mathbf{q}}{\vartheta} = \sigma$$

Constitutive relations I.

Stokes' law

$$\mathbb{S} = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \lambda(\operatorname{div} \mathbf{u})\mathbb{I}$$

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Fourier's law

$$\mathbf{q} = -\kappa \nabla \vartheta$$

Constitutive relations II.

Entropy production inequality (2^{nd} law of thermodynamics)

$$\sigma \geq \frac{1}{\vartheta}\mathbb{S} : \nabla \mathbf{u} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2}$$

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Pressure-Energy-Entropy

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$$\vartheta Ds = De + pD(\tfrac{1}{\rho})$$

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$$s(\rho, \vartheta) = \tfrac{4}{3} d \frac{\vartheta^3}{\rho} - P_\vartheta(\rho) + c_v \log \vartheta, \;\; e(\rho, \vartheta) = P_e(\rho) + d\frac{\vartheta^4}{\rho} + c_v \vartheta$$

where $P(z) = \int_1^z \frac{p(s)}{s^2} \mathrm{d}s$.

Initial and boundary conditions

Boundary condition on $\mathbf{u}$

Particles at the boundary are fixed: $\mathbf{u} = 0$ on $\partial\Omega$

Boundary condition on $\mathbf{q}$

System is thermally isolated: $\kappa(\vartheta)\nabla\vartheta \cdot \mathbf{n} = 0$ on $\partial\Omega$

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Initial conditions

$$\rho(0,\cdot) = \rho_0, \quad \rho(0,\cdot)\mathbf{u}(0,\cdot) = \mathbf{m}_0, \quad \vartheta(0,\cdot) = \vartheta_0$$

Existence results

Solutions for small data

Matsumura, Nishida, Hoff,. . .

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Variational (weak) solutions

Leray, P.-L. Lions, Feireisl, . . .

For arbitrarily large initial data one can construct solution to the problem via approximation through some easier-to-solve problems.

Variational Solutions

Definition

We say that (u, ρ, ϑ) is a variational solution for the Navier-Stokes-Fourier system on $(0, T) \times \Omega$, if

$$\begin{aligned} \partial_t b(\rho) + \operatorname{div}(b(\rho)\mathbf{u}) \; &= \\ (b(\rho) - b'(\rho)\rho) \operatorname{div} \mathbf{u} \quad &\text{in } \mathcal{D}'((0, T) \times \mathbb{R}^N), b \in BC^1[0, \infty), \end{aligned}$$

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\partial_t(\rho s) + \operatorname{div}(\rho s\mathbf{u}) - \operatorname{div}\frac{\kappa(\vartheta)\nabla\vartheta}{\vartheta} &\geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla\mathbf{u} + \frac{\kappa(\vartheta)|\nabla\vartheta|^2}{\vartheta^2}\right) \\
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&\qquad \text{in } \mathcal{D}'((0,T)\times\overline{\Omega}), \\
\partial_t \int_\Omega \frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\vartheta)\mathrm{d}x &= \int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\mathrm{d}x \text{ in } \mathcal{D}'[0,T)
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Existence of variational solutions

Review of known results

- *P.-L. Lions, 1998*: Existence for
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Some of open questions

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Some of open questions

- Ideal gas case: $p(\rho, \vartheta) = R\rho\vartheta$.
- Existence in general domains (holds for $p = p(\rho), p(\rho) \geq a\rho^{\gamma}$).

Concept for proving existence of solutions on Lipschitz domains

- Mollify the problem by adding regularizing terms (artificial viscosity in the continuity equation ($\varepsilon\Delta\rho$) and artificial pressure ($\delta\rho^{\beta}$) in the linear momentum equation).

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- **Pass with smooth domains to the Lipschitz one.**
- Pass with $\delta \to 0$.

Convergence of domains I

Lemma (Approximating sequence Ω_n)

For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ there exists a sequence of smooth domains Ω_n having uniformly Lipschitz continuous boundary with respect to n, such that $\Omega \subseteq \Omega_n$ and for any ball $B \subset \mathbb{R}^3 \setminus \Omega$ there exists n_0 such that for any $n \geq n_0$ $B \subset \mathbb{R}^3 \setminus \Omega_n$.

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Lemma (Bogovskii operator)

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists a bounded linear operator $\mathcal{B} = (B_1, B_2, B_3) : L^p(\Omega) \to W_0^{1,p}(\Omega)^3$ solving problem $\operatorname{div} \mathbf{u} = g - \frac{1}{|\Omega|} \int_\Omega g \mathrm{d}x, \mathbf{u} \in W_0^{1,p}(\Omega)$, with $1 < p < \infty$.

Furthermore, for a sequence Ω_n approximating Ω as given in the previous lemma, one can construct a uniformly bounded sequence $\mathcal{B}_n$ of Bogovskii's operators for Ω_n.

Convergence of domains II

Continuity equation

Div–Curl lemma implies that $h(\rho_n)\mathbf{u}_n \to \overline{h(\rho)}\mathbf{u}$ with $h \in BC^1[0,\infty)$, and consequently, ρ and $\mathbf{u}$ solve the (renormalized) continuity equation in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$.

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Entropy inequality and Aubin–Lions lemma implies strong convergence

$$\vartheta_n \to \vartheta \text{ in } L^1((0, T) \times \Omega).$$

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Pressure estimates

Use test function $\varphi(t,x) = \psi(t)\mathcal{B}(\rho(t,\cdot))(x)$, $\psi \in \mathcal{D}(0,T)$, for the linear momentum equation to obtain

$$\|\rho_n\|_{L^{\beta+1}((0,T)\times\Omega)} \leq c(\delta)$$

Convergence of domains III

Effective viscous pressure

Use test function $\varphi(t,x) = \psi(t)\eta(x)(\nabla\Delta^{-1})[\rho(t,\cdot)](x)$, $\psi \in \mathcal{D}(0,T)$ and $\eta \in \mathcal{D}(\Omega)$, for the linear momentum equation to obtain weak continuity of the *effective viscous pressure*

$$P_{eff} = p_e(\rho) + \vartheta p_\vartheta(\rho) + \frac{d}{3}\vartheta^4 + \delta\rho^\beta - \mathbb{S} : (\nabla\Delta^{-1}\nabla)[\rho]$$

This implies strong convergence of densities $\rho_n \to \rho$ in $L^1((0,T)\times\Omega)$.

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Linear momentum equation

Strong convergence of ρ_n and ϑ_n implies that the linear momentum equation holds for $(\rho, \mathbf{u}, \vartheta)$.

Convergence of domains IV

Energy equality

By strong convergence of ρ_n and ϑ_n, $(\rho, \mathbf{u}, \vartheta)$ satisfies the total energy equality.

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Entropy inequality

Decompose $\Omega_n = \Omega \cup (\Omega_n \setminus \Omega)$, and use weak lower semicontinuity of the terms

$$\mathbb{S}_n : \nabla \mathbf{u}_n, \text{ and } \frac{\kappa(\vartheta)|\nabla\vartheta|^2}{\vartheta^2}.$$

Results on domain dependence

- $\Omega_n \to \Omega$, $\Omega_n \subset \Omega$. Under the condition that Ω is Lipschitz, and for any $K \subset \Omega$ compact there exists n_0 such that if $n \geq n_0$, then $K \subset \Omega_n$, and moreover, $\mathrm{cap}_2(\Omega \setminus \Omega_n) \to 0$, as $n \to \infty$, solutions converge.

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- $\Omega_n \to \Omega$, $\Omega \subset \Omega_n$. Under the condition that Ω is Lipschitz, and for any ball $B \subset \mathbb{R}^3 \setminus \Omega$ there exists n_0 such that if $n \geq n_0$, then $B \subset \mathbb{R}^3 \setminus \Omega_n$, we obtain convergence of solutions.

Existence on general domains in $\mathbb{R}^3$

Main goals

- Existence of solutions for exterior Lipschitz domains in $\mathbb{R}^3$.

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Main difficulties

- Some estimates depend on the measure of the domain.
- Boundedness of the Bogovskii operator is intimately related to the Lipschitz continuity of the boundary.
- Boundedness of the Bogovskii operator may fail in the case of a non-compact boundary.

Thank you for your attention.