Large–Time Behaviour of Solutions to Quasilinear Parabolic Equations on a Half-Line

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Outline

1. Introduction
   - Statement of the Problem
   - Main Result
   - Strategy of the Proof
   - Preliminaries

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2. Bounded Solutions Bounded Away From Zero
   - Solutions Converging to Zero Form an Open Set
   - Concentrated Compactness
   - Solutions Diverging to Infinity Form an Open Set

3. Proof of the Main Result

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Consider u is a solution to

$$\left.\begin{array}{rcll} u_t - F(u_x)_x + h(u) & = & 0, & (t,x) \in (0,\infty)\times(0,\infty) \\ u(t,0) = \lim_{x\to\infty} u(t,x) & = & 0, & t \geq 0 \\ u(0,x) & = & u_0(x), & x \geq 0. \end{array}\right\} (EP)$$

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What happens as $t \longrightarrow \infty$?

- Does $u(t) \longrightarrow 0$?
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- What other phenomena can occur?

History

Convergence results

- Zelenyak
- Matano

Non-convergence results

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Convergence results

- Zelenyak
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- Feireisl, Petzeltová

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Theorem (Main Result)

Assume that the following hold:

- $h'(0) > 0, h(0) = 0$
- $\zeta_0 := \inf\{t > 0 : \int_0^t h(s)\mathrm{d}s \leq 0\} > 0,\ h(\zeta_0) < 0$

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- Set of nonnegative initial conditions is *connected*.
- Solutions tending to zero form an open set.
- Solutions with blow-up form an open set.
- There exists a bounded solution bounded away from zero.
- Apply Concentrated Compactness.

Assumptions and Notation

Basic Assumptions

- $F \in C^2(\mathbb{R}), 0 < \mu \leq F'(\cdot),$
- $h \in C^1[0, \infty), h(0) = 0, h'(0) > 0,$
- $H(t) := \int_0^t h(s) \mathrm{d}s$
- $\zeta_0 := \inf\{t > 0 : H(t) \leq 0\} > 0, h(\zeta_0) < 0.$

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Additional Assumptions

- $F(0) = 0,\ 0 < \underline{\mu} \leq F'(\cdot) \leq \overline{\mu}$
- $h(v) \geq 0, \quad v \geq s_0$

General Existence Results

Theorem (Existence of Solutions)

The problem (EP) admits for any initial datum $u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ a unique solution u belonging to

$$L^\infty((0, T \times \mathbb{R}_+) \cap L^2((0, T); W^{1,2}(\mathbb{R}_+)), \quad 0 < T < \infty.$$

Moreover, u_t, u_{xx} are Hölder continuous at any point $(t, x) \in (0, \infty) \times [0, \infty)$ and u belongs to

$$C([0, \infty); L^2(\mathbb{R}_+)) \cap C^1((0, \infty); L^2(\mathbb{R}_+)) \cap W^{1,2}_{loc}((0, \infty); W^{1,2}_0(\mathbb{R}_+))$$

General Existence Results

Theorem (Existence of Energy)

The problem (EP) admits an energy functional

$$E(v) = \int_0^\infty (I(v_x) + H(v))\mathrm{d}x, \quad v \in W_0^{1,2}(\mathbb{R}_+),$$

where $I(t) = \int_0^t F(s)\mathrm{d}s$.
The energy functional is non-increasing along any trajectory and the mapping

$$t \mapsto E(u(t))$$

is continuously differentiable at any time $t > 0$ *with*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u(t)) = -\|u_t(t)\|^2_{L^2(\mathbb{R}_+)}$$

Continuous Dependence on Initial Data

Theorem

Consider the mapping

$$\Phi_t : u_0 \mapsto u(t), \quad t \geq 0$$

where u is a solution to the problem (EP).
Then for any $t > 0$, Φ_t is continuous as a mapping

$$\Phi_t : L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \cap \{u : \mathbb{R}_+ \to \mathbb{R}_+\} \to L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+).$$

Stationary Solution on $\mathbb{R}$

Theorem (Ground State Solutions)

There exists a unique solution w_g of the problem

$$\left.\begin{array}{rcl} -F(w_x)_x + h(w) & = & 0 \quad \text{on } \mathbb{R} \\ w(0) & = & \max\{w(x) : x \in \mathbb{R}\} > 0 \\ \lim_{|x|\to\infty} w(x) & = & 0. \end{array}\right\} (SP)$$

Moreover, the following hold

$$\begin{array}{rcl} w_g(0) & = & \zeta_0 \\ w_g(\cdot) & > & 0 \\ w_g'(-x) > 0 & , & w_g'(x) < 0, \quad x > 0 \\ w_g & \in & W^{2,2}(\mathbb{R}). \end{array}$$

Shifted Stationary Solution on $\mathbb{R}$

Theorem

For any $\varepsilon > 0$ sufficiently small there exists a unique solution w to the problem

$$\left.\begin{array}{rcll} -F(w_x)_x + h(w) & = & 0 & \text{on } \mathbb{R} \\ w(0) & = & \zeta_0 + \varepsilon & \\ w_x(0) & = & 0. & \end{array}\right\} (SSP)$$

Moreover, there exist constants

$$-\infty < L_- < 0 < L_+ < \infty$$

such that

$$w(L_-) = 0 = w(L_+).$$

Stability of the Zero Solution

Lemma

The zero solution of (EP) *is locally asymptotically stable.*
This means that there exists U *– a* $L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ *neighbourhood of* $w_0 \equiv 0$ *in the set of nonnegative functions such that for any* $u_0 \in U$ *and the corresponding solution* u *we have*

$$\lim_{t \to \infty} \|u(t)\|_{W_0^{1,2}(\mathbb{R}_+)} = 0$$

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Theorem

The set of nonnegative initial conditions for which solutions converge to zero form an open set in the set of nonnegative functions with respect to the $L^2 \cap L^\infty$ *topology.*

Number of Solution's Humps

Theorem (One-Hump Solutions)

Let u_0 be a nonnegative initial condition for u, such that

- *on* $[0, \gamma]$ *is* u_0 *nondecreasing,*
- *on* $[\gamma, \infty)$ *is* u_0 *non-increasing.*

Then for any $t > 0$ *there exists* $\gamma > 0$ *such that* $u(t)$ *does so. Consequently,* $\{x > 0 : u(t, x) > \alpha\}$ *is an interval for any* $t, \alpha > 0$.

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Proof.

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Proof.

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2. Apply the zero number property to u^δ_x.
3. Apply $u^{\delta_n} \to u$ for $\delta_n \to 0$. ☐

Concentrated Compactness

Theorem (Concentrated Compactness)

Let u_0 *be nonnegative nondecreasing on* $[0, \gamma_0]$ *and non-increasing on* $[\gamma_0, \infty)$. *Let* $u^n = u(t_n)$ *be a sequence such that* $\|u^n\|_{W^{1,2}(\mathbb{R}_+)}$ *is uniformly bounded with respect to* n. *Then either*

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- $u(t)$ *converges to zero in* $W^{2,2}(\mathbb{R}_+)$ *as* $t \to \infty$, *or*
- *there exists a function* y, $\lim_{t\to\infty} y(t) = \infty$ *such that*

$$\lim_{t\to\infty} \|u(t) - w_g(\cdot - y(t))\|_{W^{2,2}(\mathbb{R}_+)} = 0.$$

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6. If $\|u(s_n)\|_{W^{1,2}(\mathbb{R}_+)} \to \infty$ then there exists σ_n such that $\|u(\sigma_n)\|_{W^{1,2}(\mathbb{R}_+)} = 2\|w_g\|_{W^{1,2}(\mathbb{R}_+)}$ is bounded.

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7. Any y^n such that $\|u(t_n) - w_g(\cdot - y^n)\|_{W^{2,2}(\mathbb{R}_+)} \to 0$ must diverge to ∞.

☐

Conditions for $W^{1,2}(\mathbb{R}_+)$ Divergence

Lemma

Let $\varepsilon > 0$ *be sufficiently small. Then there exists* $L > 0$ *such that if* $u_0 \geq \zeta_0 + \varepsilon$ *on some interval of length* L*, then*

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3. Compare u and w as solutions on interval I: $u(t) \geq w, t \geq 0$.
4. $\sup_{x>0} w_g(x) = \zeta_0 < \zeta_0 + \varepsilon = \sup_{x\in I} w(x) \leq \sup_{x>0} u(t, x)$ for any $t \geq 0$. □

Conditions for $W^{1,2}(\mathbb{R}_+)$ Divergence

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For any $\varepsilon > 0$ *sufficiently small there exists* $L' > 0$ *such that if* $u_0 \geq \zeta_0 - \varepsilon$ *on an interval of length* L'*, then*

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1. Show that for any $L > 0, \varepsilon > 0$ there is $L' > 0$ such that if $u_0 = (\zeta_0 - \varepsilon)\chi_{[a,b]}$, $|b - a| \geq L'$, then for some $t_0 > 0$ it holds $u(t_0, \cdot) \geq \zeta_0 + \varepsilon$ on interval of length L.

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2. Compare $u(t)$ to the solution of $v'(t) + h(v(t)), v(0) = \zeta_0 - \varepsilon$ and estimate the difference in $L^2(\mathbb{R}, e^{-|x-c|}), c = \frac{a+b}{2}$.

☐

Solutions Diverging to Infinity Form an Open Set

Theorem

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Proof.

1. Let $\|u(t)\|_{W^{1,2}(\mathbb{R}_+)}$ diverge. For $J(t) = \{x : u(t,x) > \zeta_0 - \varepsilon\}$ we have
$$\lim_{t\to\infty} |J(t)| = \infty.$$

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2. Use continuous dependence on initial data for initial condition v_0 near u_0 in $L^2 \cap L^\infty$-norm. ☐

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5. Take solution $\tilde{u}$ to (EP) with $\tilde{u}(0) = u(t_0)$.

□

Topics for Improvement

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- Domains in higher dimensions.

That's All. . .

Thank you for your attention.