Projective modules over universal enveloping algebras

September 27, 2011
Intro

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- $R$-module means a right module over $R$
- A module $P$ is said to be *projective* if it is isomorphic to a direct summand of $R_R^{(\kappa)}$
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- A module $P$ is said to be projective if it is isomorphic to a direct summand of $R_{R}^{(\kappa)}$

**Problem:** For a (specified) ring give a classification of projective modules. By a theorem of Kaplansky every projective module is a direct sum of countably generated ones.
Projective modules are relevant in homological algebra, geometry and direct sum decompositions problems in module theory.

Theorem (Swan) Let $X$ be a compact topological space. The category of real vector bundles on $X$ is equivalent to the category of finitely generated projective modules over $\mathbb{C}[X]$.

Add $(M)$ consists of direct summands of $M$ for some $\mathcal{A}$.

Theorem (Dress) Let $M$ be a finitely generated module over $\mathbb{R}$. The category $\text{Add}(M)$ is equivalent to the category of projective modules over $\text{End}_{\mathbb{R}}(M)$.
Why?

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**Theorem**

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$\text{Add}(M)$ consists of direct summands of $M^{(\kappa)}$ for some $\kappa$.

**Theorem**

*Dress* Let $M$ be a finitely generated module over $R$. The category $\text{Add}(M)$ is equivalent to the category of projective modules over $\text{End}_R(M)$.
Noetherian setting

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Typical situation over noetherian rings is that finitely generated projective modules are harder to classify than nonfinitely generated ones. Evolution of so called Serre’s problem (1955):

- In 1957 Serre proved that finitely generated projectives over $k[x_1, \ldots, x_n]$ are stably free.
- In 1963 Bass proved that nonfinitely generated projectives over $k[x_1, \ldots, x_n]$ are free.
- In 1976 Suslin and Quillen independently proved that all projective modules over $k[x_1, \ldots, x_n]$ are free.
Noetherian setting cont.

On the other hand sometimes there is a nice connection between properties of the ring $R$ and the structure theory for nonfinitely generated projective modules. Let us discuss the case of an integral group ring of a finite group.
Noetherian setting cont.

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**Theorem**

(Swan, Akasaki, Linnell) If $G$ is a finite group then $G$ is solvable if and only if every nonfinitely generated projective module over $\mathbb{Z}G$ is free.
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On the other hand sometimes there is a nice connection between properties of the ring $R$ and the structure theory for nonfinitely generated projective modules. Let us discuss the case of an integral group ring of a finite group.

**Theorem**

*(Swan, Akasaki, Linnell)* If $G$ is a finite group then $G$ is solvable if and only if every nonfinitely generated projective module over $\mathbb{Z}G$ is free.

If $P$ is a projective module $\text{Tr}(P) = \sum_{f: P \to R} f(P)$. This is an idempotent ideal in $R$ (*trace ideal of $P$*) and $P\text{Tr}(P) = P$.

**Theorem**

*(Bass)* If $R$ is a connected commutative noetherian ring then every nonfinitely generated projective module over $R$ is free.
Fair-sized modules

We say that a ring $R$ satisfies (*) if every chain of two-sided ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ such that $I_{k+1}I_k = I_{k+1}$ stabilizes.

If $R$ is a (left and right) noetherian ring satisfying (*) then countably generated projective modules can be classified by pairs $(I, \overline{P})$, where $I$ is an idempotent ideal of $R$ and $\overline{P}$ is a finitely generated projective module over $R/I$. 

Theorem (Puninski) Let $n \not\equiv 0 \pmod{8}$ be a square-free integer. Then every torsion-free module over $\mathbb{Z}[\!\! [p] \!\!]$ is a direct sum of finitely generated modules if and only if $n \equiv 1 \pmod{8}$.
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- countably generated projective module $P$ corresponds to $(I, P/PI)$, where $I$ is the smallest ideal of $P$ such that $P/PI$ is finitely generated.
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- $(0, P)$ corresponds to a finitely generated projective $R$-module $P$. 
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- $(R, 0)$ corresponds to $R_R^{(\omega)}$
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- $(I, \overline{P})$ are decomposable whenever $0 \neq I$.
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Theorem

(Puninski) Let $n \neq 0$ be a square-free integer. Then every torsion-free module over $\mathbb{Z} [\sqrt{n}]$ is a direct sum of finitely generated modules if and only if $n \neq 1 \mod 8$. 
Lie Algebras

Let $k$ be a field, an algebra $(L, [\cdot, \cdot])$ is called a Lie algebra if

- $[\cdot, \cdot]$ is bilinear
- $[x, x] = 0$ for every $x \in L$
- $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ for every $a, b, c$

A representation of $L$ is a Lie algebra homomorphism of $L$ to $gl_n(k)$

Any associative algebra give a Lie structure via $[a, b] := ab - ba$.

Define $L^{(1)} = L$, $L^{(2)} = [L, L], \ldots, L^{(i+1)} = [L^{(i)}, L^{(i)}], \ldots$

A Lie algebra $L$ is said to be solvable if there exists $i \in \mathbb{N}$ such that $L^{(i)} = 0$.

A radical of a Lie algebra $L$ is the greatest solvable ideal in $L$.

A Lie algebra $L$ is called semisimple if has zero radical.

A Lie algebra $L$ is called simple if $[L, L] \neq 0$ and there are no nontrivial ideals of $L$. 
Basic structure theorems for Lie algebras

**Theorem**

(Cartan) If $L$ is a finite dimensional semisimple Lie algebra over a field of characteristic zero, then $L = L_1 \oplus \cdots \oplus L_n$, where every $L_i$ is an ideal of $L$ which is a simple Lie algebra.

**Theorem**

(Levi) Let $B$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $R$ be the radical of $B$. Then $B$ contains a finite-dimensional semisimple subalgebra $L$ such that $B = L \oplus R$. 
Universal enveloping algebras

Let $L$ be a Lie algebra with basis $\{x_1, \ldots, x_n\}$ over $k$. Let

$$[x_i, x_j] = \sum_{k=1}^{n} c_{i,j,k} x_k.$$ 

The algebra $U(L) = k\langle x_1, \ldots, x_n \rangle / \langle x_i x_j - x_j x_i - \sum_{k=1}^{n} c_{i,j,k} x_k \rangle$ is called the universal enveloping algebra of $L$.

- Monomials $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ form a basis of $U(L)$, so $L$ is contained in $U(L)$. 
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- If $A$ is an associative $k$-algebra and $\varphi : L \to A$ a linear map such that $\varphi([a, b]) = \varphi(a)\varphi(b) - \varphi(b)\varphi(a)$ then there is a unique extension of $\varphi$ to a homomorphism $\Phi : U(L) \to A$. 


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- The category of representations of \( L \) is equivalent to the category of finite-dimensional right \( U(L) \)-modules.
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- The category of representations of $L$ is equivalent to the category of finite-dimensional right $U(L)$-modules.
- $U(L)$ is a noetherian domain.
Universal enveloping algebras

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- Krull dimension of $U(L)$ is at most $\dim_k(L)$. 
Some results on finitely generated projectives

Theorem

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- (Stafford) If $P$ is a finitely generated projective module over $U(L)$ of rank bigger than $\dim_k(L)$, then $P$ is free. In particular, $\text{Tr}(P) = U(L)$ for every nonzero finitely generated projective $U(L)$-module.
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- (Stafford) If $[L, L] \neq 0$ then $U(L)$ contains a right ideal which is stably free but not free.
Solvable case

Theorem

Let \( k \) be a field of characteristic 0 and let \( L \) be a solvable Lie algebra of finite dimension over \( k \). If \( I_1, I_2, \ldots \) is a sequence of ideals in \( U(L) \) such that \( I_{k+1} I_k = I_{k+1}, k \in \mathbb{N} \). Then either \( I_k = U(L), k \in \mathbb{N} \) or there exists \( l \) such that \( I_l = 0 \).
Solvable case

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**Corollary**

If $L$ is a solvable Lie algebra of finite dimension over a field of characteristic zero then every non-finitely generated projective $U(L)$-module is free.

If $[L, L] = 0$ we get the Bass’ result for $k[x_1, \ldots, x_n]$. 
Simple case

**Theorem**

(Weyl) Let $L$ be a semisimple Lie algebra of finite dimension over a field of characteristic zero. Then every $U(L)$-module of finite dimension is completely reducible. In particular, if $I$ is an ideal of finite codimension over $U(L)$, then $U(L)/I$ is semisimple artinian.
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**Corollary**

(Kraft, Small, Wallach) If $L$ is as above then every ideal in $U(L)$ of finite codimension is idempotent.

Put $R = U(sl_2(\mathbb{C}))$. For every $k$ there is a simple $R$-module $M_k$ of dimension $k$. Every $I_k = \text{Ann}(M_k)$ is an ideal of codimension $k^2$ therefore is idempotent. By a result of Whitehead every $I_k$ is a trace ideal of a projective module which cannot contain a finitely generated direct summand.
Problems in $U(sl_2(\mathbb{C}))$

- One can show that fair-sized theory cannot work here. There exists a countably generated projective module $P$ such that \( \{ I \subseteq U(sl_2(\mathbb{C})) \mid P/PI \text{ is finitely generated} \} \) has not the least element.
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- There are at least uncountably many countably generated projective $U(\mathfrak{sl}_2(\mathbb{C}))$-modules up to isomorphism.
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- Let $R = U(\mathfrak{sl}_2(\mathbb{C}))$ and $\hat{R}$ its completion in linear topology induced by ideals of finite codimension. Let $P, Q$ be countably (finitely) generated projective $R$-modules such that $P \otimes_R \hat{R} \simeq Q \otimes_R \hat{R}$. Does it mean $P \simeq Q$?
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- Is there a way how to classify projective modules over $U(\mathfrak{sl}_2(\mathbb{C}))$?
End.

Thanks for your attention.