Projective modules are determined by their radical factors

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Abstract

We show that two infinitely generated projective modules are isomorphic whenever they have isomorphic factors modulo their Jacobson radical. Some applications of the result to semilocal rings with indecomposable non-finitely generated projective modules are given.

1 Introduction

Using a projective cover argument, one can show that two finitely generated projective modules are isomorphic if and only if they have isomorphic factors modulo their Jacobson radicals. This well known result can be used to get information about finitely generated projective modules over semilocal rings. For example, Fuller and Shutters [7] proved that over any semilocal ring there are only finitely many indecomposable finitely generated projective modules up to isomorphism.

The aim of this note is to prove that arbitrary projective modules \( P, Q \) are isomorphic whenever they have isomorphic factors modulo their Jacobson radical. Let us briefly recall some related results achieved so far.

For some results saying when an infinitely generated projective module is free see [2] and [12]. In fact, we are interested when it is not the case. Beck [3] proved that if \( P \) is a projective right \( R \)-module and \( P/\text{rad}(P) \) is a free right \( R/J(R) \)-module, then \( P \) is a free \( R \)-module. Later, Gruson (see the appendix of [12]), proved that any free base of \( P/\text{rad}(P) \) can be lifted to a free base of \( P \). Jøndrup [11] used an idea of Bergman to prove that if \( P, Q \) have isomorphic radical factors, then \( P \) can be embedded to \( Q \) and \( Q \) can be embedded to \( P \). A generalization by Facchini, Herbera and Sakhajev [6] gives that if \( P, Q \) are projective modules and there exists a pure monomorphism from \( P/\text{rad}(P) \) to \( Q/\text{rad}(Q) \), then there is a pure monomorphism from \( P \) to \( Q \).

We prove the result in the title and then we give several immediate corollaries for projective modules over semilocal rings. For example, we show that there are at most countably many indecomposable projective modules over a semilocal ring. As a bit more sophisticated application we show how to use knowledge of objects in \( \text{Add} \) of a uniserial module to give a classification of right projective modules over an endomorphism ring of a

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uniserial module. Also we get answers to some problems Puninski posed in [13] easily. The last part of this note deals with an example of a semilocal ring having a projective module that is not a direct sum of indecomposable modules. Let us stress that over commutative semilocal rings the situation is much easier. Indeed, it follows from Hinohara [9, 10] that over commutative weakly noetherian rings (hence also over commutative semilocal rings) every projective module is a direct sum of finitely generated modules.

All basic results about the Jacobson radical can be found, for example, in [1]. Unless otherwise stated, we work inside the category of right modules over a (fixed) associative ring $R$ with unit. We denote $J(R)$ the (Jacobson) radical of $R$ and $\text{rad}(M)$ the (Jacobson) radical of the module $M$. If $P$ is a projective module, then $\text{rad}(P) = PJ(R)$. We call $P/\text{rad}(P)$ the radical factor of $P$.

## 2 The result

If $P,Q$ are projective modules and $\pi_P: P \to P/\text{rad}(P)$, $\pi_Q: Q \to Q/\text{rad}(Q)$ are the canonical projections, then for any homomorphism $\overline{f}: P/\text{rad}(P) \to Q/\text{rad}(Q)$ there exists a homomorphism $f: P \to Q$ such that $\pi_Q f = \overline{f} \pi_P$. We say that $f$ is a lift of $\overline{f}$. The idea we are going to use in the next lemma is essentially described in [11].

**Lemma 2.1** Let $P, Q$ be countably generated projective modules. Suppose that $\overline{f}: P/\text{rad}(P) \to Q/\text{rad}(Q)$ and $\overline{g}: Q/\text{rad}(Q) \to P/\text{rad}(P)$ are mutually inverse isomorphisms. Let $f: P \to Q$ be any lift of $\overline{f}$ and let $X \subseteq P$ be a finite set. Then there exists a lift $g: Q \to P$ of $\overline{g}$ such that $g f(x) = x$ for any $x \in X$.

**Proof.** Let $P', Q'$ be projective modules such that $P \oplus P'$ and $Q \oplus Q'$ are countably generated free modules. It is possible to suppose $f': P' \to Q'$ and $g': Q' \to P'$ are mutually inverse morphisms. (In fact, we can suppose $P' = Q' = R^\omega$ because of the Eilenberg’s trick.) Let $g_0$ be any lift of $\overline{g}$ and let us fix some free base $Y = \{e_1, e_2, \ldots\}$ of $P \oplus P'$. Consider the homomorphism $h = (g_0 + g') \circ (f \oplus f') : P \oplus P' \to P \oplus P'$. For any $e \in P \oplus P'$ is $h(e) = e \in \text{rad}(P \oplus P')$. Let $n \in \mathbb{N}$ be such that any element of $X$ can be expressed as a combination of $e_1, \ldots, e_n$. We claim there is an endomorphism $h': P \oplus P' \to P \oplus P'$ lifting the identity on $P \oplus P'/\text{rad}(P \oplus P')$ such that $h'(e_i) = e_i$ for any $i = 1, \ldots, n$. In order to see this, express $h$ as a column-finite matrix $A$ (the $i$-th column is formed by coordinates of $h(e_i)$ determined by the base $Y$). Let $m \geq n \in \mathbb{N}$ be such that first $m$ columns of $A$ have non-zero values only in first $m$ rows. Let $B$ be a $m \times m$ matrix given by the top left corner of $A$. Consider $B$ as an element of the ring $M_m(R)$. Then $B \in 1 + J(M_m(R))$ is an invertible matrix and its inverse $C$ is also an element of $1 + J(M_m(R))$. Replacing the top left $m \times m$ corner in the identical $\mathbb{N} \times \mathbb{N}$ matrix by $C$ we obtain a column-finite matrix $A'$ that represents desired endomorphism $h'$ with respect to the base $Y$.

Let $\pi_P: P \oplus P' \to P$ be the canonical projection and let $\iota_P: P \to P \oplus P'$ be the canonical inclusion. Then we can put $g = \pi_P h' \iota_P g_0$. □

**Lemma 2.2** Let $P, Q$ be countably generated projective modules such that $\overline{f}: P/\text{rad}(P) \to Q/\text{rad}(Q)$ is an isomorphism. Then there exists a lift of $\overline{f}$ which is an isomorphism.

**Proof.** Let $\{p_0, p_1, \ldots\}$ be generators for $P$ and let $\{q_0, q_1, \ldots\}$ be generators for $Q$. Let $\overline{g}: Q/\text{rad}(Q) \to P/\text{rad}(P)$ an inverse of $\overline{f}$. We are going to define homomorphisms $f_i: P \to Q$, $g_i: Q \to P$ and finite sets $P_i \subseteq P$, $Q_i \subseteq Q$ for any $i \in \mathbb{N}_0$ as follows:
Put $P_0 = \{ p_0 \}$ and let $f_0$ be any lift of $\overline{f}$. Suppose $P_i, f_i$ were defined, define $Q_i, g_i$ by $Q_i = f_i(P_i) \cup \{ q_0, \ldots, q_i \}$ and let $g_i$ be a lift of $\overline{f}$ such that $g_if_i(x) = x$ for any $x \in P_i$.

Suppose $Q_i, g_i$ were defined, define $P_{i+1} = g_i(Q_i) \cup \{ p_0, \ldots, p_{i+1} \}$ and let $f_{i+1}$ be a lift of $\overline{f}$ such that $f_{i+1}g_i(x) = x$ for any $x \in Q_i$.

Observe that $P_i \subseteq g_i(Q_i) \subseteq P_{i+1}$. If $p \in P_i$, then $p = g_i f_i(p)$ and $f_{i+1}(p) = f_{i+1}g_i f_i(p) = f_i(p)$ since $f_i(p) \in Q_i$. Therefore $f_{i+1}(P_i) = f_i(P_i)$. Thus we can define $f: P \to Q$ by $f(p) = f_i(p)$ if $p \in \langle P_i \rangle$.

Suppose that $f(p) = 0$. Then $p \in \langle P_i \rangle$ for some $i \in \mathbb{N}$. But then $0 = g_i f_i(p) = g_i f_i(p) = p$. Therefore $f$ is mono. In order to see that $f$ is epi, just observe $f(P_{i+1}) \supseteq Q_i$. Finally it remains to prove that $f$ is a lift of $\overline{f}$. But this is obvious since all $f_i$'s are lifts of $\overline{f}$. □

**Theorem 2.3** Let $P, Q$ be projective modules such that $\overline{f}: P/\text{rad}(P) \to Q/\text{rad}(Q)$ is an isomorphism. Then there is an isomorphism $f: P \to Q$ which is a lift of $\overline{f}$.

**Proof.** By the theorem of Kaplansky, there are decompositions $P = \oplus_{i \in I} P_i$ and $Q = \oplus_{j \in J} Q_j$ such that the modules $P_i, Q_j, i \in I, j \in J$ are countably generated. It is well known that $\text{rad}(P) = \oplus_{i \in I} \text{rad}(P_i)$, $\text{rad}(Q) = \oplus_{j \in J} \text{rad}(Q_j)$. As in the proof of [4, Theorem 2.50] we find an ordinal $\kappa$ and sets $I_\lambda \subseteq I, J_\lambda \subseteq J, \lambda \leq \kappa$ such that

(i) $I_0 = \emptyset = J_0$

(ii) $I_\lambda \subseteq I_\lambda, I_\lambda \subseteq I_\kappa$ for any $\lambda' < \lambda \leq \kappa$

(iii) $I_\lambda = \cup_{\lambda' < \lambda} I_{\lambda'}$ and $J_\lambda = \cup_{\lambda' < \lambda} J_{\lambda'}$, if $\lambda \leq \kappa$ is limit

(iv) If $\lambda < \kappa$, then $|I_{\lambda+1} \setminus I_\lambda| \leq \omega$ and $|J_{\lambda+1} \setminus J_\lambda| \leq \omega$.

(v) $I = I_\kappa, J = J_\kappa$

(vi) $\overline{f}(\oplus_{i \in I_\lambda} P_i/\text{rad}(P_i)) = \oplus_{j \in J_\lambda} Q_j/\text{rad}(Q_j)$

For any $\lambda \leq \kappa$ let $P_\lambda = \oplus_{i \in I_\lambda} P_i$, $Q_\lambda = \oplus_{j \in J_\lambda} Q_j$ and let $\overline{f}_\lambda, \overline{Q}_\lambda$ be the corresponding radical factors. Observe that $\overline{f}_\lambda$ gives an isomorphism of $\overline{P}_\lambda$ and $\overline{Q}_\lambda$.

By induction on $\lambda \leq \kappa$ we construct isomorphisms $f_\lambda: P_\lambda \to Q_\lambda$ such that $f_{\lambda+1}$ extends $f_\lambda$ for any $\lambda < \kappa$ and $f_\lambda$ is a lift of $\overline{f}_\lambda$ for any $\lambda \leq \kappa$. We put $f_0 = 0$.

If $\lambda < \kappa$ and $f_\lambda$ has been defined, we define $f_{\lambda+1}$ as follows: Let $P' = \oplus_{i \in I_{\lambda+1} \setminus I_\lambda} P_i$, $Q' = \oplus_{j \in J_{\lambda+1} \setminus J_\lambda} Q_j$ and let $\overline{P}', \overline{Q}'$ be their radical factors. So $P_{\lambda+1} = P_\lambda + P'$ and $Q_{\lambda+1} = Q_\lambda + Q'$. Consider the restriction $\overline{f}|_{\overline{P}}: \overline{P} \to \overline{Q}$ and put $\overline{\sigma} = \overline{\pi} \overline{f}|_{\overline{P}}$ and $\overline{\beta} = \overline{\pi} \overline{f}|_{\overline{P}}$. By (vi), $\overline{\beta}$ is an isomorphism, and $P', Q'$ are countably generated according to (iv), therefore, by Lemma 2.2, there is an isomorphism $\overline{\beta}: P' \to Q'$ lifting $\overline{\beta}$. Since $P'$ is projective, there exists $\alpha: P' \to Q_\lambda$ lifting $\overline{\sigma}$. If we put $f_{\lambda+1} = f_\lambda \oplus (\alpha + \beta)$, we can check that $f_{\lambda+1}$ is an isomorphism extending $f_\lambda$ and lifting $\overline{f}_{\lambda+1}$.

If $\lambda$ is limit, and $f_\lambda$ has been defined for every $\lambda' < \lambda$, we put $f_\lambda = \cup_{\lambda' < \lambda} f_{\lambda'}$. By induction, $f_\lambda: P_\lambda \to Q_\lambda$ is an isomorphism lifting $\overline{f}_\lambda$.

Finally, $f = f_\kappa$ is the desired isomorphism. □

Some well known results about projective modules can be seen also as corollaries of Theorem 2.3.

**Corollary 2.4**
(i) Any nonzero projective module has a maximal submodule.

(ii) Let $R$ be a local ring. Then any projective module is free.

(iii) Let $R$ be a semiperfect ring, let $S_1, \ldots, S_n$ be representatives of simple modules and let $P_i$ be a projective cover of $S_i$ for any $1 \leq i \leq n$. Then any projective module can be uniquely decomposed as a direct sum of copies of $P_1, \ldots, P_n$.

We hope that Theorem 2.3 is a tiny step toward understanding of projective modules over semilocal rings. Recall that a ring is semilocal if $R/J(R)$ is semisimple, thus the radical factor of a projective module over a semilocal ring is semisimple $R$ (or $R/J(R)$)-module. Facchini and Herbera [5] gave a description of direct sum decompositions of finitely generated projective modules over a semilocal ring. In particular, it is proved that for any semilocal ring $R$ there exists a semilocal hereditary ring $R'$ such that $R$ and $R'$ have the same decomposition theory of finitely generated projective modules. As we shall see this is not true for arbitrary projective modules because any projective module over a hereditary ring is a direct sum of finitely generated modules. However, some well known properties of finitely generated projective modules over a semilocal ring can be generalized.

**Corollary 2.5** Let $R$ be a semilocal ring. If $P, Q$ are projective right $R$-modules, then the following are equivalent

(i) $P \simeq Q$

(ii) There exist epimorphisms $f: P \to Q$ and $g: Q \to P$.

(iii) There exist pure monomorphisms $f: P \to Q$ and $g: Q \to P$.

Moreover, $P^n \simeq Q^n$ implies $P \simeq Q$ for any $n \in \mathbb{N}$.

**Proof.** Since $P/\operatorname{rad}(P)$ and $Q/\operatorname{rad}(Q)$ are semisimple, each of (ii), (iii) implies $P/\operatorname{rad}(P) \simeq Q/\operatorname{rad}(Q)$. Now Theorem 2.3 applies. □

Using Kaplansky’s theorem once again we obtain the following

**Corollary 2.6** Let $R$ be a semilocal ring. Then there are at most countably many pairwise non-isomorphic indecomposable projective modules.

We do not know an example of a semilocal ring having infinitely many non-isomorphic indecomposable projective modules (recall that over a semilocal ring there are only finitely many non-isomorphic indecomposable finitely generated projective modules.) Observe that there would be only finitely many indecomposable projective modules over a semilocal ring if the following was true: If $P, Q$ are projective modules and $P/\operatorname{rad}(P)$ is a direct summand of $Q/\operatorname{rad}(Q)$, then $P$ is a direct summand of $Q$. Unfortunately, this is not true. Gerasimov and Sakhajev [8] gave an example of a semilocal ring $R$ which possesses an infinitely generated projective module $P$ such that $P/\operatorname{rad}(P)$ is finitely generated and hence a direct summand of $F/\operatorname{rad}(F)$, where $F$ is a suitable finitely generated free module. Of course, $P$ cannot be a direct summand in $F$. Similar phenomena will occur in sections 4 and 5.

Our last corollary uses a technique of Sakhajev to give information about the structure of projective modules having radical factor cyclic. Recall that a sequence $a_1, a_2 \ldots \in R$ is called a right $a$-sequence if $a_i = a_{i+1} a_i$ for any $i \in \mathbb{N}$. 

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Corollary 2.7 Let \( P \) be projective \( R \)-module such that \( P/\text{rad}(P) \) is cyclic. Then there exists \( r \in R \) and \( p_1, p_2, \ldots \in P \) such that \( P = \sum_{i \in \mathbb{N}} p_i R \) and \( p_{i+1} r = p_i \) for any \( i \in \mathbb{N} \).

Proof. We shall use the idea of [13, Fact 3.1]. Observe that \( P \) is countably generated. Moreover, \( P \) can be considered as a pure right ideal of \( R \) by [6, Proposition 6.1]. Take \( p \in P \) such that \( (pR + \text{rad}(P))/\text{rad}(P) = P/\text{rad}(P) \). Since \( P \) is pure in \( R_R \), there exists \( q \in P \) such that \( qp = p \). By assumption, there is \( t \in R \) and \( j \in \text{rad}(P) = J(R) \cap P \) such that \( q = pt + j \). Now, \( p = qp = (pt + j)p \) and \( pt = (pt)^2 + jpt \) follows. Since \( j \in J(R) \), the element \( u = (1 - j) \) is invertible and \( upt = (pt)^2 \). For any \( i \in \mathbb{N} \) put \( x_i = u^{-i}ptu^{-i-1} \). Then \( x_{i+1} x_i = u^{-i-1}(pt)^2u^{-i-1} = x_i \), which is \( x_1, x_2, \ldots \) is a right \( a \)-sequence. As proved in [13, Fact 3.1], \( Q = \sum_{i \in \mathbb{N}} x_i R \) is a pure right ideal and hence projective. Since, by purity, \( \text{rad}(P) = P \cap J(R) \), \( \text{rad}(Q) = Q \cap J(R) \), the canonical projection \( \pi: R \to R/J(R) \) induces embeddings of \( P/\text{rad}(P) \) and \( Q/\text{rad}(Q) \) into \( R/J(R) \). Obviously, \( \pi(P) = pR + J(R)/J(R) \). The definition of \( x_i \)'s gives \( \pi(Q) \subseteq pR + J(R)/J(R) \) and since \( \pi(x_1)p = \pi(p) \), the equality \( \pi(P) = \pi(Q) \) holds. Thus \( P/\text{rad}(P) \cong Q/\text{rad}(Q) \) and \( P \cong Q \) by Theorem 2.3. For any \( i \in \mathbb{N} \) put \( q_i = x_i u^{-i+1} \). Observe that \( q_1, q_2, \ldots \) generate \( Q \) and \( q_{i+1} pt = q_i \) for any \( i \in \mathbb{N} \).

\( \square \)

Remark 2.8 Observe that the element \( r \in R \) can be chosen in the trace ideal of \( P \). Since \( P^r R = P \), we infer that the trace ideal of \( P \) is \( R R \).

3 Comparing \( \text{Add}(M_R) \) and \( \text{Proj-End}_R(M_R) \)

We are going to investigate the relation between objects of \( \text{Add}(M_R) \) (i.e., direct summands of modules that are direct sums of copies of \( M \)) and \( \text{Proj-End}_R(M_R) \) (i.e., projective right modules over the endomorphism ring of \( M \)), where \( M \) is a nonzero \( R \)-module. Probably we shall reinvent a wheel but we were not able to find a convenient reference. One could simply say that the tensor product commutes with direct sums but we need to be more explicit.

Let \( M \) be a nonzero right module over a ring \( R \) and let \( I \) be a non-empty set. Let \( S \) denote the endomorphism ring of \( M \). Consider a direct sum decomposition of the free right \( S \)-module \( A \oplus B = S^{(l)} \), let \( \iota_A: A \to S^{(l)} \), \( \iota_B: B \to S^{(l)} \) be canonical inclusions. Applying the tensor product functor \( - \otimes S: \text{Mod-S} \to \text{Mod-R} \), we get \( \text{Im} (\iota_A \otimes M) \oplus \text{Im} (\iota_B \otimes M) = S^{(l)} \otimes M \). The module \( S^{(l)} \otimes M \) is isomorphic to \( M^{(l)} \) via the isomorphism \( \varphi: S^{(l)} \otimes M \to M^{(l)} \) given by \( \varphi((s_i)_{i \in I} \otimes m) = (s_i(m))_{i \in I} \). Denote \( A' = \text{Im} \varphi \circ (\iota_A \otimes M) \), \( B' = \text{Im} \varphi \circ (\iota_B \otimes M) \) and observe that \( A' \oplus B' = M^{(l)} \). Let \( \iota_{A'}: A' \to M^{(l)} \), \( \iota_{B'}: B' \to M^{(l)} \), \( \pi_{A'}: M^{(l)} \to A' \), \( \pi_{B'}: M^{(l)} \to B' \) be canonical injections and projections given by this decomposition. Any \( i \in I \) also gives the canonical injection \( \iota_i: M \to M^{(l)} \) and the canonical projection \( \pi_i: M^{(l)} \to M \). Fix an arbitrary \( j \in I \) and consider the element \( e_j = (\delta_{i,j})_{i \in I} \in S^{(l)} \), where \( \delta_{i,j} = 1 \) if \( i = j \) and \( \delta_{i,j} = 0 \) if \( i \neq j \). There exist unique \( (a_i)_{i \in I} \in A \) and \( (b_i)_{i \in I} \in B \) such that \( e_j = (a_i)_{i \in I} + (b_i)_{i \in I} \). Then \( \iota_j(m) = \varphi(e_j \otimes m) = (a_i(m))_{i \in I} + (b_i(m))_{i \in I} \). Note that \( (a_i(m))_{i \in I} \in A' \) and \( (b_i(m))_{i \in I} \in B' \). Therefore \( \pi_{i \iota_{A'}} \pi_{A'} \iota_j = a_i \) and \( \pi_{i \iota_{B'}} \pi_{B'} \iota_j = b_i \) for any \( i \in I \). Therefore a direct sum decomposition of \( S^{(l)} \) in \( \text{Mod-S} \) induces a decomposition of \( M^{(l)} \). Unfortunately, not every decomposition of \( M^{(l)} \) can be constructed in such a way because the module \( \pi_{A'}(\iota_i(M)) \) has a finite support for any \( i \in I \). Thus we have defined a map \( \Phi \) which assigns a decomposition of \( M^{(l)} \) to every decomposition of \( S^{(l)} \).
Now let us consider the decomposition \( A' \oplus B' = M^{(I)} \) such that \( \pi_{A'}(\iota_i(M)) \) has finite support for any \( i \in I \), that is for any \( i \in I \) there exist only finitely many \( j \in I \) such that \( \pi_{A'} \pi_{A_i} \neq 0 \). We shall say that \( A', B' \) form a finite support decomposition of \( M^{(I)} \). For any \( i,j \in I \) let us denote \( a_{j,i} = \pi_{A'} \pi_{A_i} \in S \) and \( b_{j,i} = \pi_{B'} \pi_{B_i} \in S \). Since \( \pi_{A'}(\iota_i(M)) \) (and hence also \( \pi_{B'}(\iota_i(M)) \)) has finite support, we have \( a_i = (a_{j,i})_{j \in I} \in S^{(I)} \) and \( b_i = (b_{j,i})_{j \in I} \in S^{(I)} \). Put \( A = \sum_{i \in I} a_i S \subseteq S^{(I)} \), \( B = \sum_{i \in I} b_i S \subseteq S^{(I)} \). Since \( a_i + b_i = e_i \), \( A + B = S^{(I)} \). Suppose that there exists nonzero \( x \in A \cap B \). Then there are \( s_i \in S, i \in I \), almost all of them equal zero, such that \( x = \sum_{i \in I} a_i s_i \). Take some \( j \in I \) and \( m \in M \) such that \( \sum_{i \in I} a_{j,i} s_i(m) \) is nonzero. Observe that \( x' = \sum_{i \in I} (a_{j,i}(s_i(m))) = \sum_{i \in I} \pi_{A'}(\iota_i(s_i(m))) \) is a nonzero element of \( A' \). By our assumption there are \( t_i \in S, i \in I \), almost all of them zero, such that \( x = \sum_{i \in I} b_i t_i \). By the same arguments as above, we infer \( x' \in B' \), a contradiction. Therefore \( A + B = S^{(I)} \). Now we have defined a map \( \Psi \) that assigns a decomposition of \( S^{(I)} \) to every finite support decomposition of \( M^{(I)} \). It can be easily verified that \( \Phi \) and \( \Psi \) are mutually inverse.

Now we can summarize these observations in

**Proposition 3.1** Let \( M \) be a nonzero right \( R \)-module, let \( S = \text{End}_R(M) \), and let \( I \) be a nonempty set. Put \( C = \{(A,B) \mid A \subseteq S^{(I)}, B \subseteq S^{(I)}, A \oplus B = S^{(I)}\} \) and \( D = \{(A',B') \mid A', B' \text{ form finite support decomposition of } M^{(I)}\} \). The maps \( \Phi: C \to D \) and \( \Psi: D \to C \) are mutually inverse bijections.

If \( A \oplus B = A_1 \oplus B_1 = S^{(I)} \) are two decompositions of \( S^{(I)} \), and \( A \simeq A_1 \), then \( A' \simeq A_1' \), where \( (A', B') = (A(B), (A_1', B_1')) = \Phi((A_1, B_1)) \). This is because \( \Phi \) is "carried" by a functor. But we do not have an analogy to this statement in the opposite direction, so the classification of the projective \( S \)-modules can be different from the classification of objects of \( \text{Add}(M) \) that arise from finite support decompositions of \( M^{(I)} \). But observe that projective modules should be more complex than objects in \( \text{Add}(M) \) given by finite support decompositions (tensor product can make non-isomorphic modules isomorphic) but, of course, the later class may be quite different from \( \text{Add}(M) \).

## 4 Projective modules over the endomorphism ring of a biuniform module

Now we apply these general concepts to the particular case of the endomorphism ring of a biuniform module. Recall that a module \( M \) is called biuniform if it is nonzero, \( M \) is not a sum of its two proper submodules and any two nonzero submodules of \( M \) have a non-trivial intersection. A module is said to be uniserial if its lattice of submodules is a chain. Obviously, any nonzero uniserial module is biuniform. Let \( S = \text{End}_R(M) \). By [4, Theorem 9.1] \( I = \{f \in S \mid f \text{ is not mono}\} \) and \( K = \{f \in S \mid f \text{ is not epi}\} \) are two-sided ideals. If \( I, K \) are comparable by inclusion, then \( S \) is local and hence all projective modules are free. Therefore we shall consider only the opposite case, \( I, K \) incomparable and, by [4, Theorem 9.1], \( I, K \) are the only maximal right ideals of \( S \). Then \( S \) is semilocal and \( S/J(S) \simeq S/I \times S/K \). Note that simple \( S \)-modules \( S/I, S/K \) cannot be isomorphic. Following [13], we shall use the following notation: Let \( P \) be a countably generated projective \( S \)-module. Then \( P/\text{rad}(P) \simeq S/I^{(k)} \oplus S/K^{(l)} \) for some \( 0 \leq k, l \leq \omega \). Since \( k, l \)
are uniquely determined by $P$, we can define $\dim(P) = (k, l)$ (the dimension of $P$). In particular, $\dim(S) = (1, 1)$, hence, if $P$ is a free $S$-module, then $\dim(P) = (k, k)$ for some $0 \leq k \leq \omega$.

The following lemma is easy to prove see for example [17, Lemma 2.2].

**Lemma 4.1** Let $U_i, i \in I$ be a family of biuniform modules. Suppose $A \oplus B = \oplus_{i \in I} U_i$. If $A$ is nonzero, then there are $i, j \in I$ such that $\pi_{j \cdot A}^\ast \pi_{A \cdot i}$ is a monomorphism.

The following lemma answers [13, Question 8.1].

**Lemma 4.2** Let $P$ be a countably generated projective $S$-module. If $\dim(P) = (0, k)$, then $P = 0$.

**Proof.** We can suppose that $P \oplus Q = S^{(\omega)}$ for some module $Q$. By [13, Remark 2.3] $P/\text{rad}(P) \simeq P/P I \oplus P/P K$, hence our assumption is equivalent to $P = PI$. Thus if $(s_i)_{i \in \omega} \in P$, then none of $s_i$’s is a monomorphism. Suppose $P \neq 0$. Applying the map $\Phi$ of Proposition 3.1 to $(P, Q)$ we get a decomposition $P' \oplus Q' = M^{(\omega)}$. Since $\Phi$ is mono, $P'$ is nonzero. Moreover, we saw that the endomorphisms $\pi_{j \cdot P'}^\ast \pi_{P' \cdot i}$, $i, j \in \omega$ are given as coordinates of elements in $P$. Hence none of these endomorphisms is a monomorphism, a contradiction to Lemma 4.1. □

**Proposition 4.3** Let $P$ be a countably generated projective $S$-module. If $\dim(P) = (k, l)$, then $k \geq l$.

**Proof.** Suppose there exists a countably generated projective $S$-module $P$ such that $\dim(P) = (k, l)$ and $k < l$. Then $k < \omega$. Observe that $\dim(S) = (1, 1)$. Since $S$ is a finitely generated projective module, there exists $Q$ such that $P \simeq S^k \oplus Q$. Because the dimension is additive, $\dim(Q) = (0, l')$, $l' \neq 0$. This contradicts Lemma 4.2. □

Let us recall a part of the main result of [13].

**Fact 4.4** [13, Theorem 4.3] Let $M$ be a biuniform $R$-module, $S = \text{End}_R(M)$. Then the following are equivalent:

(i) There is a monomorphism $f \in S$ and an epimorphism $g \in S$ such that $gf = 0$.

(ii) There exists a countably generated projective $S$-module $P$ such that $\dim(P) = (1, 0)$.

Observe that our results give a classification of projective $S$-modules in case $S$ satisfies the equivalent conditions of this theorem. Namely, if $P$ is a module of dimension $(1,0)$, then all projective right $S$-modules are isomorphic to $P^{(X)} \oplus S^{(Y)}$.

Right now we are not able to say more if $M$ is an arbitrary biuniform module. In case $M$ is uniserial one can complete the classification using the following:

**Fact 4.5** Let $M$ be a nonzero uniserial module such that $\text{End}_R(M)$ is not local. Assume there is a decomposition $A \oplus B = M^{(\omega)}$, $A \neq 0$ such that $\pi_{j \cdot A}^\ast \pi_{A \cdot i} \in \text{End}_R(M)$ is not an epimorphism for any $i, j \in \omega$. Then there are a monomorphism $f: M \to M$ and an epimorphism $g: M \to M$ such that $gf = 0$. 7
Remark 4.8 can be shown that any self-small uniserial module is quasi-small.

Proof. Suppose that $gf \neq 0$ for every monomorphism $f: M \rightarrow M$ and for every epimorphism $g: M \rightarrow M$. Then first two paragraphs in [17, proof of Proposition 2.7] applied to the decomposition $A \oplus B = M^{(\omega)}$ give a contradiction. □

Proposition 4.6 Let $M$ be a nonzero uniserial module. Suppose that $gf \neq 0$ for any monomorphism $f: M \rightarrow M$ and any epimorphism $g: M \rightarrow M$. Then any projective $S$-module is free.

Proof. By a classical result of Kaplansky any projective module over a local ring is free (we reproved it in Corollary 2.4). Thus we can suppose that $\text{End}_R(M)$ is not local. Let $P$ be a countably generated projective $S$-module that is not free. By Proposition 4.3 we can assume that $\dim(P) = (k,0)$, where $k \neq 0$. Let $Q$ be an $S$-module such that $P \oplus Q = S^{(\omega)}$. Applying $\Phi$ we obtain a decomposition $P' \oplus Q' = M^{(\omega)}$, where $P'$ is nonzero. Since $\dim(P) = (k,0)$, this decomposition satisfies the assumption of Fact 4.5. That implies the existence of a monomorphism $f: M \rightarrow M$ and an epimorphism $g: M \rightarrow M$ such that $gf = 0$. □

Thus we reached the classification of projective modules over the endomorphism ring of a uniserial module $U$ that is quite similar to that of modules in $\text{Add}(U)$.

Theorem 4.7 Let $U$ be a nonzero uniserial module $R$-module and let $S = \text{End}_R(U)$. Then every right projective $S$-module is free if and only if $gf \neq 0$ for every monomorphism $f: U \rightarrow U$ and every epimorphism $g: U \rightarrow U$. In the opposite case there is a countably (but not finitely) generated projective $S$-module $P$ such that $P/\text{rad}(P)$ is simple and every right projective $S$-module is isomorphic to $P^{(X)} \oplus S^{(Y)}$.

Recall that a module $U$ is called self-small if for any homomorphism $f: U \rightarrow U^{(\omega)}$ there exists a finite set $I \subseteq \omega$ such that the image of $f$ is contained in $U^I$. A module $U$ is said to be quasi-small if for any family $M_i, i \in I$ of modules such that $U$ is a direct summand of $\oplus_{i \in I} M_i$, there exists a finite set $I_0 \subseteq I$ such that $U$ is isomorphic to a direct summand of $\oplus_{i \in I_0} M_i$. For example, any finitely generated module is self-small and quasi-small. It can be shown that any self-small uniserial module is quasi-small.

Remark 4.8 Let $U$ be a uniserial module such that there are $f \notin I$ and $g \notin K$ such that $gf = 0$. Suppose that there is $u \in U$ such that $h(u) \neq 0$ for any $h \notin K$. By [17, Theorem 1.1], there exists a unique uniserial module $V \neq U$ such that $V$ is a direct summand of $U^{(\omega)}$. The module $V$ is not quasi-small. In this case all objects of $\text{Add}(U)$ are isomorphic to direct sums of copies of $U$ and $V$. It can be proved that $\text{Hom}_R(U,V)$ is a projective right $S = \text{End}_R(U)$-module of dimension $(1,0)$. Moreover, the Hom - tensor adjunction induces an equivalence of $\mathcal{K}$ and $\mathcal{L}$, where $\mathcal{K}$ is the full subcategory of $\text{Add}(U)$ given by modules of finite Goldie dimension and $\mathcal{L}$ is the full subcategory of $\text{Proj}-S$ given by projective modules with finitely generated radical factor. An example of a cyclic uniserial module $U$ of required property can be found in [15] but we do not know whether $U$ can be chosen such that $\text{Hom}_R(U, -)$ does not commute with direct sums.

Question 4.9 Is there a uniserial module $U$ satisfying the following:

(i) $U$ is quasi-small.
(ii) $U$ is not self-small.

(iii) There exist a monomorphism $f: U \to U$ and an epimorphism $g: U \to U$ such that $gf = 0$.

Observe that (ii) implies $U$ is a countably but not finitely generated module. We do not know the answer even if (ii) is replaced by this weaker condition.

5 Pure projective modules over an exceptional chain ring

In this section we are going to describe a case, where a projective module that is not possible to decompose as a direct sum of indecomposable modules occurs.

Using Theorem 2.3 we can finish the classification of pure projective modules over an exceptional chain ring (see [14, Conjecture 5.9]). As most of the work was already done in [14], we shall be as brief as possible but we will follow an abstract approach introduced in [18]. An interested reader is advised to see [4], [14], [16] and [18] for details.

Let $R$ be a ring and let $T, U$ be finitely generated uniserial modules such that

(i) The endomorphism ring of $T$ is local.

(ii) There exists a uniserial module $V \nsubseteq U$ such that $V$ is a direct summand of $U^{(\omega)}$. Such $V$ is unique up to isomorphism by [17, Theorem 1.1].

(iii) There exists a module $W$ such that $U \oplus T^{(\omega)} \cong V \oplus W$.

Let $M = T \oplus U$ and let $S = \text{End}_R(M)$. Since $M$ is finitely generated, categories $\text{Add}(M)$ and $\text{Proj}-S$ are equivalent. Let us denote $P_1 = \text{Hom}_R(M, T)$, $P_2 = \text{Hom}_R(M, U)$, $P_3 = \text{Hom}_R(M, V)$ and $P_4 = \text{Hom}_R(M, W)$ (take some $W$ satisfying (iii), we shall see that it is in fact unique). Now we want to understand radical factors of $P_3$. Namely, set $S_1 := P_3/\text{rad}(P_3)$ is simple. Further $\text{End}_S(P_2) \cong \text{End}_R(U)$, $U$ cannot have the endomorphism ring local by (ii), therefore $P_2$ has exactly 2 maximal submodules $X_1, X_2$ such that $S_2 := P_2/X_1$ and $S_3 := P_2/X_2$ are not isomorphic. Namely, set $X_1 = \{ f: T \oplus U \to U \mid f|_U \text{ is not monic} \}$ and $X_2 = \{ f: T \oplus U \to U \mid f|_U \text{ is not epic} \}$. Observe that an arbitrary $f: U \to U$ is not a monomorphism (resp. not an epimorphism) if and only if $\text{Im} \text{ Hom}_R(M, f) \subseteq X_1$ (resp. $\text{Im} \text{ Hom}_R(M, f) \subseteq X_2$).

Since $T$ is not a direct summand of $U$, $S_1, S_2, S_3$ are pair-wise non-isomorphic simple modules and $S/J(S) \cong S_1 \oplus S_2 \oplus S_3$. Again, we shall write $\dim(P) = (a, b, c)$ if $a, b, c$ are cardinals such that $P/\text{rad}(P) \cong S_1^{(a)} \oplus S_2^{(b)} \oplus S_3^{(c)}$. There exists a split monomorphism $\nu: V \to U^{(\omega)}$ such that $\pi_i \nu: V \to U$ is not a monomorphism for any $i > 0$ ($\pi_i$ stands for the canonical projection $U^{(\omega)} \to U$), see [4, Proof of Proposition 9.30] for details. Then it is easy to check that $\dim(P_3) = (0, 1, 0)$. Finally, we derive $\dim(P_4) = (\omega, 0, 1)$ easily from $P_1^{(\omega)} \oplus P_2 \cong P_3 \oplus P_4$. Note that $W$ is indeed described by (i),(ii),(iii) uniquely up to isomorphism. Let us summarize our calculations:

$\dim(P_1) = (1, 0, 0), \dim(P_2) = (0, 1, 1), \dim(P_3) = (0, 1, 0), \dim(P_4) = (\omega, 0, 1)$.

Now we claim that any projective $S$-module is isomorphic to a direct sum of copies of $P_1, P_2, P_3$ and $P_4$. Let $Q$ be a countably generated projective module of dimension $(a, b, c)$. 

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If $a = \omega$, then $P_3^{(b)} \oplus P_4^{(c)} \simeq Q$. If $a < \omega$, there is an $S$-module $Q'$ such that $Q = Q' \oplus P_1^a$ because $P_1$ is finitely generated. If $b \geq c$, then $Q'$ is a direct sum of copies of $P_2$ and $P_3$. If $b < c$, then, since $P_2$ is finitely generated, there exists a projective $S$-module $Q''$ such that $\dim(Q'') = (0, 0, d)$. Since $Q'' \oplus P_3^{(d)} \simeq P_2^{(d)}$ there would be a module in $\text{Add}(U)$ that is not a direct sum of copies of $V$ and $U$, a contradiction to Theorem 4.7. This proves the claim and we are ready to classify pure projectives over some very strange rings.

Recall that a ring $R$ is called a chain ring if $R R, R R$ are (left and right) uniserial $R$-modules. Following [16], a chain ring is said to be exceptional if it has exactly 3 two-sided ideals 0, $J(R), R, R$ is prime and contains zero divisors. Henceforth, let $R$ be an exceptional coherent chain ring. By [16, Lemma 3.5] for any $0 \neq r, s \in J(R)$ modules $R/rR$ and $R/sR$ are isomorphic and any pure projective $R$-module is isomorphic to a direct summand of a direct sum of copies of $R, R/rR$ for some (any) $0 \neq r \in J(R)$. Let $U = R/rR, T = R_R$. Then (i) follows since $T$ is projective and uniserial, (ii) holds by [14, Lemma 4.2] and (iii) holds by [14, Lemma 4.3]. As remarked above, pure projective modules over $R$ are exactly objects of $\text{Add}(U \oplus T)$ and categories $\text{Add}(U \oplus T)$ and $\text{Proj-End}_R(U \oplus T)$ are equivalent, therefore we have

**Theorem 5.1** Let $R$ be an exceptional chain coherent ring. Then any pure projective module is isomorphic to a direct sum of copies of $T, U, V, W$.

**Remark 5.2** It was noted in [18] that $W$ is not a direct sum of uniserial modules, but in fact $P_4$ cannot be written as a direct sum of indecomposable modules and so does not $W$. Indeed, by [14, Proposition 4.5] any direct sum decomposition of $W$ is of the form $W \oplus T^{(\delta)}$, where $0 \leq \delta \leq \omega$. This statement now follows easily from Theorem 5.1 and remains valid in the more abstract context from the beginning of this section.

**References**


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