Lecture notes on the P-ideal dichotomy

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Abstract

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1 Introduction

Todorcevic introduced combinatorial statements that have the form of an ideal dichotomy. Let I be an ideal of countable sets (which means that $I \subset [S]^{\leq\omega}$ for some set S is closed under subsets and finite unions; it is convenient to assume that $[S]^{<\omega} \subseteq I$). We say that $X \subseteq S$ is "out of I" (or orthogonal to I) iff $I \cap [X]^{\omega} = \emptyset$. We say that X is "inside I" iff $[X]^{\leq\omega} \subseteq I$. In plain words: out of I means that no infinite subset of X is in I, and inside I means that all countable subsets of X are in I. Again, X is out of I if the restriction of I on X is the ideal of finite sets, and X is inside I if the restriction of I on X is the ideal of *all* countable subsets. The simplest form of the dichotomy for an ideal of countable sets I over an uncountable S is the following statement:

> There is an uncountable $X \subseteq S$ which is
>
> 1. inside I, or
> 2. out of I.

[1]Some parts of these lectures were presented in the set-theory seminar, Jerusalem, January-February 2007.

We shall prove the consistency of this combinatorial statement.

Why I is required to be an ideal? Let I consists of all countable subsets of ω_1 of order type $\geq \omega + \omega$. (So I is not closed under subsets.) There is no uncountable set out of I and no set in I. Another example is where I is closed under subsets, but not under finite unions. I consists of all subsets of ω_1 of order type $\leq \omega + \omega$. Again, there is no uncountable set out of I, and no uncountable set inside I.

The ideal of subsets of ω_1 of order-type less than ω^2 is not a P-ideal, that is an increasing sequence has no upper bound in the almost inclusion relation. The dichotomy does not hold for this ideal.

Two strong forms of ideal dichotomy were introduced by Todorcevic and proved to follow from the Proper Forcing Axiom PFA.

If I is any ω_1 generated ideal over a set S then:

form 1 : Either there is a decomposition $S = \bigcup_{i \in \omega} S_i$ where each S_i is inside I, or there is an uncountable subset of S that is out of I.

form 2 : Either there is a decomposition $S = \bigcup_{i \in \omega} S_i$ where each S_i is out of I, or there is an uncountable subset of S that is inside I.

First Form

We prove that PFA implies the first form of the dichotomy. Clearly there is no loss of generality in assuming that I consists only of countable sets (just consider $I \cap [S]^{\leq \aleph_0}$). Another simple argument shows that we may assume that S contains all singletons.

Since I is ω_1-generated, we may assume $S = \omega_1$ and I is generated by $\{X_\alpha \mid \alpha < \omega_1\}$ where $X_\alpha \subseteq \alpha$ for every $\alpha \in \omega_1$.

Assume that there is no decomposition of S into countably many sets that are inside I, and we shall show that some proper poset forces an uncountable set outside of I.

Definition of the poset. $p \in P$ iff $p = (x_p, d_p, N^p)$ is such that the following hold.

1. $N^p = \{N_0^p, \ldots, N_{k-1}^p\}$ is a finite set of countable elementary substructures of $H(\aleph_2)$ enumerated such that $N_i^p \in N_{i+1}^p$ for every $i < k$.

$I \in N^p_0$. ($H(\aleph_2)$ is the structure consisting of all sets of cardinality hereditarily less than $\aleph_2$.)

2. $x_p \in [\omega_1]^{<\omega}$ is "separated" by N^p. This means that for every two points in x_p, say $\alpha < \beta$, there is a model N^p_i such that $\alpha < N^p_i \cap \omega_1 < \beta$. For notational purposes, it is convenient to assume that x_p contains $k+1$ points. So that if $x_p = \alpha_0 < \cdots < \alpha_k$, then we have $\alpha_0 < N^p_0 \cap \omega_1 < \alpha_1 < N^p_1 \cdots N^p_{k-1} < \alpha_k$.

3. For every α in x_p and structure N^p_i not containing α (we say α is "above N_i") α does not belong to any set in N^p_i that is inside I: $\alpha \notin \bigcup\{X \mid X \in N^p_i \text{ is inside } I\}$.

4. $d_p \in [\omega_1]^{<\omega}$. The role of x_p is to generically develop an uncountable set outside of I, and the role of d_p is to fix $x_p \cap X_\alpha$ for $\alpha \in d_p$.

Define $p \leq q$ (q extends p; is more informative) iff

1. $x_p \subseteq x_q$, $d_p \subseteq d_q$, and $N^p \subseteq N^q$.

2. For every $\alpha \in d_p$, $x_p \cap X_\alpha = x_q \cap X_\alpha$.

For every $p \in P$ and $\gamma \in \omega_1$ there is an extension q of p such that $x_q \setminus \gamma \neq \emptyset$. Proof. We may assume that the largest (last) model of p contains x_p and d_p. (If not then add such a countable elementary substructure of $H(\aleph_2)$.) Let N^p_{k-1} be that largest model of p and assume that γ is in that model as well. If $\omega_1 \setminus \bigcup\{X \mid X \in N^p_i \text{ is inside } I\}$ is countable, then ω_1 is a countable union of sets that are inside I. Since this is not the case, we can find an ordinal $\alpha \in \omega_1 \notin N^p_i$ and add it to x_p.

Clearly, it is always possible to add an ordinal to d_α.

So we can define ω_1 dense sets in P ensuring that the filter G that intersects them indeed produces a set $X = \bigcup\{x_p \mid p \in G\}$ that is uncountable and is out of I.

The main point is to prove properness of P so that the proper forcing axiom can be applied. Take any cardinal κ, consider the structure of all sets of cardinality hereditarily less than κ and let $M \prec H(\kappa)$ be countable, and $p_0 \in M \cap P$ an arbitrary condition. Define $p = p_0 + (M \cap H(\aleph_2))$ (that is, add $(M \cap H_{\aleph_2})$ to the set of models of p_0, this results in an ϵ-sequence and hence a condition that extends p_0). We shall prove that p is an M-generic condition over P. For this we must prove that if p' is any extension of p and

D is a dense set $D \in M$, then p is compatible with a member of $D \cap M$. We may assume that already $p' \in D$ (or else extend p' into D). The following lemma concludes the argument (apply it to p').

Lemma 1.1 *Suppose that $p = (x_p, d_p, N^p)$ is a condition in P (where $N^p = (N_0^p, \ldots, N_{k-1}^p)$), and $M \prec H(\kappa)$ is a countable elementary substructure such that $M \cap H(\aleph_2) = N_i^p$ for some $i < k$. Suppose moreover that $D \in M$ is dense open in P, and $p \in D$. Then p is compatible with a condition in $D \cap M$.*

Proof. Recycling the name p_0, define the M-"lower part" of p: $p_0 = p \restriction M = (x_p \cap M, d_p \cap M, (N_0^p, \ldots, N_{i-1}^p))$ be the lower part of p. So $p_0 \in M$ since $N_{i-1}^p \in N_i^p$. We shall define an extension q of p_0 in $D \cap M$ and prove that it is compatible with p. The only possible obstacle is that X_α for some $\alpha \in d_p$ (out of M) intersects $x_q \setminus x_{p_0}$.

Suppose that $x_p = \alpha_0 < \alpha_1 < \cdots < \alpha_k$. Define

$$E = \{x \in [\omega_1]^{k+1} \mid x \text{ is an end-extension of } x_{p_0} \text{ and for some } d, N : (x, d, N) \in D \text{ extends } p_0\}.$$

E is non-empty since $x_p \in E$. Clearly $E \in M \cap H_{\aleph_2}$ because it is definable from parameters in M. We have $x_{p_0} = \alpha_0 < \cdots < \alpha_i$.

By downwards induction on $\ell = k, k-1, \ldots, i$ define the formula $\varphi_\ell(b_{i+1}, \ldots, b_\ell)$, where $b_{i+1} < \cdots < b_\ell$ are ordinal free-variables, which says the following:
there exists $Y_{\ell+1}$ not inside I such that for every $a_{\ell+1} \in Y_{\ell+1}$
there exists $Y_{\ell+2}$ not inside I such that for every $a_{\ell+2} \in Y_{\ell+2}$
$\vdots$
there exists Y_k not inside I such that for every $a_k \in Y_k$

$$(\alpha_0, \cdots, \alpha_i, b_{i+1}, \cdots, b_\ell, a_{\ell+1}, \cdots, a_k) \in E.$$

For $\ell = i$, $\varphi_i()$ is the sentence which we want to get:
there exists Y_{i+1} not inside I such that for every $a_{i+1} \in Y_{i+1}$
there exists Y_{i+2} not inside I such that for every $a_{i+2} \in Y_{i+2}$
$\vdots$
there exists Y_k not inside I such that for every $a_k \in Y_k$ $(\alpha_0, \cdots, \alpha_i, a_{i+1}, \cdots, a_k) \in E$.

Claim 1.2 *For every $i \leq \ell \leq k$, $\varphi_\ell(\alpha_{i+1}, \ldots, \alpha_\ell)$. In particular $\varphi_i()$.*

For the proof of the claim (1.2), work your way from the last member of x_p downwards, using the following lemma.

Lemma 1.3 *If $N \prec H_{\aleph_2}$ is countable, and $b \in \omega_1 \setminus N$ is not member of any set in N that is inside I, if $\phi(x, y)$ is any formula such that $\phi(b, a)$ holds (where $a \in N$), then $\{x \in \omega_1 \mid \phi(x, a)\}$ is not inside I.*

(Since this set is definable in N it belongs to N, and as it contains b it cannot be inside I.)

Now we can get an extension of p_0 in $D \cap M$ that is compatible with p. Let $X = \bigcup\{X_\alpha \mid \alpha \in d_p\}$. Then $X \in I$. Using $\varphi_i()$ we get $Y_{i+1} \in M$ not inside I and we pick $a_{i+1} \in Y_{i+1} \setminus X$ (as $Y_{i+1} \in M$ is a countable set, we have $Y_{i+1} \subset M$ and hence $a_{i+1} \in M$). Then we get $Y_{i+2} \in M$ not inside I, and we can pick $a_{i+2} \in Y_{i+2} \setminus X$ in M. We continue in this way until we finish with some $a_k \notin X$ so that $(\alpha_0, \cdots, \alpha_i, a_{i+1}, \cdots, a_k) \in E \cap M$. By definition of E we get an extension q of p_0 in $D \cap M$ of the form $q = (\langle \alpha_0, \cdots, \alpha_i, a_{i+1}, \cdots, a_k \rangle, d, N)$. The choice of the a_j's out of X ensures that q and p are compatible.

Second Form

Next, we show that the second form follows from PFA:

> If I is any ω_1 generated ideal over a set S then: Either there is a decomposition $S = \bigcup_{i \in \omega} S_i$ where each S_i is out of I, or there is an uncountable subset of S that is inside I.

Assume that I is ω_1 generated ideal of countable sets, and no uncountable subset of ω_1 is inside of I. We may assume that I consists of countable subsets of ω_1. We shall find a decomposition of ω_1 into countably many sets that are out of I (orthogonal to I). Assume PFA, and take an arbitrary uncountable set $X \subseteq \omega_1$. Since X does not contain an uncountable subset that is inside I, it surely is not the countable union of sets that are inside I. Hence by the First Form applied to X, X contains an uncountable subset that is outside of I. Thus, every uncountable subset of ω_1 contains an uncountable subset that is out of I. Now a c.c.c forcing poset introduces a partition $\omega_1 = \bigcup_{i \in \omega} S_i$ where every S_i is out of I.

Theorem 1.4 *Let I be an $\aleph_1$ generated ideal of countable subsets of ω_1 such that every uncountable subset of ω_1 contains an uncountable subset that is out* *of I. Then there is a c.c.c poset that forces a partition of ω_1 into countably many subsets that are out of I.*

Proof. The conditions in P are pairs $p = (f_p, d_p)$ where f_p is a finite function from ω_1 to ω, and d_p is a finite subset of ω_1. Define $p \leq q$ iff f_q and d_q extend f_p and d_p, and, for every $\alpha \in d_p$, for every k in the range of f_p, $f_p^{-1}\{k\} \cap X_\alpha = f_q^{-1}\{k\} \cap X_\alpha$.

If we let G be a generic filter over P, then $f_G : \omega_1 \to \omega$, and $\omega_1 = \bigcup\{d_p \mid p \in G\}$. It is clear that for every $n \in \omega$ and $\alpha < \omega_1$ $(f_G^{-1}\{n\}) \cap X_\alpha$ is finite. So that $f_G^{-1}\{n\}$ is indeed out of I. So it remains to prove that P is c.c.c.

For this aim, let $\{p_\alpha \mid \alpha < \omega_1\}$ be given, $p_\alpha = (f_\alpha, d_\alpha)$. Assume that the domains of f_α and the d_α form Δ systems (and in particular all f_α have the same cardinality k). Applying the property that every uncountable subset of ω_1 contains an uncountable subset that is out of I, we obtain uncountable $E \subset \omega_1$ such that for every $j < k$ the jth components of p_α for $\alpha \in E$ is out of I (or is a single point if in the core). Let O be the finite union of these out of I sets. So O is out of I, and hence for every p_α the intersection of the X_β for $\beta \in d_\alpha$ with O is finite. Another Δ argument yields conditions that are pairwise compatible. q.e.d

Here is an application due to Todorcevic of the Simple Dichotomy theorem.

Theorem 1.5 *PFA implies that there no S-spaces. In fact, the simple dichotomy for $\aleph_1$-generated ideals implies that there are no S-spaces.*

Proof. Recall the definition: An S-space is a regular, hereditarily separable, but not hereditarily Lindelof topological space. To prove that no such space exists (under the dichotomy), suppose that X is a regular topological space which is not hereditarily Lindelof and we shall prove that X is not hereditarily separable. Since X is not hereditarily Lindelof, X has a subspace $S = \{x_\alpha \mid \alpha < \omega_1\}$ such that every initial part $S_\delta = \{x_\alpha \mid \alpha \leq \delta\}$ is open in S (i.e. S is "right-separated"). We consider the subspace topology on S and shall find a subset of S which is not separable. Since S is regular, each x_α has an open neighborhood U_α with closure $\overline{U}_\alpha \subset S_\alpha$. These countable closed sets generate an ideal I. By the dichotomy, there is an uncountable set $D \subset S$ which is either "in" or "outside" of I. If D is in, then every countable subset E of D is in I, which means that it is covered by a countable closed set, and hence E is not dense in D. If D is outside of I, then D has a finite intersection with every set in I. So in particular the intersection of D with every U_α is finite. As S is a Hausdorff space, D is discrete (and therefore not separable). q.e.d

We give another application due to Todorcevic. Let $(E, \leq)$ be a partially ordered set. We say $X \subseteq E$ is an antichain if no two members of X are $\leq$ comparable. We say X is directed if every finite subset of X is bounded (not necessarily in X, but if the bound is required to be in X then it suffices to require that any two are bounded). Milner and Prikry (The cofinality of a partially ordered set) proved that every poset with no uncountable antichain is the union of continuum many directed sets. Todorcevic outline the following proof. First, take a cofinal subset of E that is well-founded (define it by induction). We may thus assume that E is well founded (any directed set in a cofinal subset of E generates a directed initial segment of the original poset). Observe that every uncountable subset of E contains an infinite chain (or else it would contain an uncountable antichain). Take a maximal collection of distinct maximal directed initial segments of E. The union is all of E, or else some $e \in E$ is outside, but then the initial segment $\leq e$ generated by e can be extended to a maximal directed initial segment which leads to a contradiction. So we must prove that this maximal collection has size $2^{\aleph_0}$. Suppose for a contradiction that $\{A_i \mid i < (2^{\aleph_0})^+\}$ is a collection of distinct maximal directed initial segments. In particular, they are $\subseteq$ incomparable initial segments. Going to the complements B_i of A_i we get an easy contradiction. These are final segments incomparable in $\subseteq$. Claim: If every antichain in E is countable, if $\{B_i \mid i < (2^{\aleph_0})^+\}$ is a collection of final segments, then for some $i < j$ $B_j \subseteq B_i$. This can be obtained with the Erdos-Rado partition theorem. Namely, if $i < j$ pick minimal member of B_j not in B_i and note its index (the minimal members of B_j form an antichain which is countable).

Now, under PFA every poset E in which every antichain is countable is a union of countably many directed sets. As above we may assume that E is well founded. Let I be the ideal of countable sets that are finite union of directed sets. By the dichotomy, either E is a countable union of sets inside the ideal (which is good), or else there is an uncountable set out of I. That is, no infinite directed set, and in particular no infinite chain. Hence an uncountable antichain.

PID = the P-ideal dichotomy

We say that $I \subseteq [S]^{\leq\omega}$ is a P-ideal if it is a non-trivial ideal over a set S (containing all singletons of S) and whenever $X_n \in I$ for $n \in \omega$ then there is $X \in I$ such that $X_n \subseteq^* X$ for all n. We say that X is an "almost cover" for $\{X_n \mid n \in \omega\}$. ($A \subseteq^* B$ is almost inclusion, which means that $A \setminus B$ is finite.)

It may seem that dichotomy for P-ideals is weaker than dichotomy for arbitrary ideals described above. Yet, it turns out that for P-ideals the dichotomy can be obtained consistently with CH ([1]) and the restriction of that dichotomy, namely that the ideal is ω_1 generated, can be removed ([3]).

If I is an ideal over S and $S_0 \subset S$, then the restriction of I on S_0 is the ideal $\{X \cap S_0 \mid X \in I\}$. If I is a P-ideal, then its restriction is again a P-ideal.

To begin with, assume the PFA. We prove that dichotomy of Form 2 holds for any P-ideal I over S:

Theorem 1.6 *Assume PFA. Let I be a P-ideal over an arbitrary (uncountable) set S. Either there is a decomposition $S = \bigcup_{i\in\omega} S_i$ where each S_i is outside of I, or there is an uncountable subset of S that is inside I. This property is known as the PID.*

The proof is by induction on the cardinality of S. So assume that $|S| = \mu$ and that the dichotomy holds for any P-ideal over a set of cardinality smaller than μ. Suppose that S is not a countable union of sets that are out of I and we shall find an uncountable set that is inside I. If there is an uncountable subset $S_0 \subseteq S$ of smaller cardinality than μ which is not a countable union of sets that are out of I then by the inductive hypothesis there is an uncountable set inside of I and we are done. Hence we may assume that every uncountable subset of S of smaller cardinality is indeed a countable union of sets out of I. We may assume $S = \mu$; that is, the ideal is over the cardinal μ itself. The minimality of μ implies that its cofinality is $> \omega$

We say that $K \subseteq I$ is cofinal in I if it is cofinal in the almost inclusion ordering $\subseteq^*$. That is, for every $X \in I$ there is $Y \in K$ such that $X \subseteq^* Y$.

The following lemma will be needed later on.

Lemma 1.7 *Suppose I is a P-ideal and $A \in I$ is countable infinite, and for every $a \in A$ we have some $X(a) \in I$. If K is cofinal in I, then there are $Y \in K$ and $a \in A$ such that $X(a) \subseteq^* Y$ and $a \in Y$.*

Proof. Since I is a P-ideal, there is some $Z \in I$ such that $A \cup \bigcup\{X(a) \mid a \in A\} \subseteq^* Z$. As K is cofinal, there is some $Y \in K$ such that $Z \subseteq^* Y$. Since A is infinite, we can find some $a \in A \cap Y$. We also get $X(a) \subseteq^* Y$ as required. q.e.d

Let P be the poset of all pairs $p = (a_p, H_p)$ where $a_p \in I$ (so a_p is countable), and H_p is a countable collection of cofinal subsets of I. Define $p \leq q$ iff $a_p \subseteq a_q$, $H_p \subseteq H_q$, and the following condition holds.

> For every $K \in H_p$, if $e = a_q \setminus a_p$ then $\delta(e, K) = \{X \in K \mid e \subseteq X\} \in H_q$.

Note that $\leq$ is indeed transitive.

Let $p = (a_p, H_p)$ be a condition. If e is any finite set disjoint from a_p, define $p + e$ as the pair $(a_p \cup e, H')$ where $H' = H \cup \{\delta(e, K) \mid K \in H_p\}$. Clearly $p + e$ is a condition iff each $\delta(e, K)$ for $K \in H_p$ is cofinal in I. In case $p + e$ is a condition, it is an extension of p. If p is any condition then a *pre-extension of* p is a pair (a, H) such that $a_p \subseteq a$, $H_p \subseteq H$ is a countable collection of cofinal in I sets, and for every $K \in H_p$, $\delta(a \setminus a_p, K)$ is cofinal in I (but is not necessarily in H). If (a, H) is a pre-extension of p, then (a, H') is an extension of p, where $H' = H \cup \{\delta(a \setminus a_p, K) \mid K \in H_p\}$.

Lemma 1.8 *Every condition $p \in P$ has, for any ordinal $\gamma < \mu$, an extension q so that a_q contains an ordinal above γ.*

Since μ has uncountable cofinality, and as a proper forcing would not change the cofinality of μ to ω, the properness of P (proved below) will imply that the generic filter generates an uncountable set inside of I.

To prove the lemma suppose on the contrary that every $\alpha > \gamma$ cannot be added to a_p (i. e. $p + \{\alpha\}$ is not a condition). Then there is a reason $K(\alpha) \in H_p$ such that $\delta(\{\alpha\}, K(\alpha))$ is not cofinal. Thus there exists some $X(\alpha) \in I$ so that no set in $\delta(\{\alpha\}, K(\alpha))$ almost includes $X(\alpha)$. That is, no set in $K(\alpha)$ that contains α also almost includes $X(\alpha)$.

Since H_p is countable, this yields a decomposition of $\mu \setminus \gamma$ into countably many classes, namely for every $K \in H_p$ we have the class C_K of those α such that $K = K(\alpha)$. But C_k is out of I (by Lemma 1.7), and the ordinals below $\gamma + 1$ have a countable decomposition into out-of-I sets by the inductive assumption. So μ is a countable decomposition into out-of-I sets which is a contradiction. q.e.d

The main point is to prove that P is proper. For this let M be countable elementary substructure of some large H_κ. Find $X_M \in I$ that almost covers every set in $I \cap M$ (a countable collection, and I is a P-ideal). Starting with some given $p_0 \in M \cap P$, we are going to define an increasing and M-generic sequence $p_i = (a_i, H_i) \in P \cap M$ with $a_i \subset X_M$, and we plan to define $p_\omega = (\bigcup_{i\in\omega} a_i, \bigcup_{i\in\omega} H_i)$ aiming that this condition is a pre-extension of every p_i. The fact that $\bigcup_{i\in\omega} a_i \in I$ is ensured by the demand that $a_i \subset X_M$. This is the main difficulty of the construction, to get that $a_i \subset X_M$ and at the same time to ensure that the sequence is M-generic. The following lemma will help us with this difficulty.

Lemma 1.9 *Suppose $p_0 = (a_0, H_0) \in P \cap M$ is an arbitrary condition, $D \in M$ is dense in P, and X almost covers every member of $I \cap M$. Then some extension q of p in $D \cap M$ is such that $a_q \setminus a_p \subset X$.*

Proof. We say that $Y \in I$ is "bad" iff for some finite $F \subset Y$ for every extension $q \in D$ of p_0, $a_q \setminus a_0 \not\subseteq Y \setminus F$. We claim that some $Y \in I$ is not bad. Suppose on the contrary that every $Y \in I$ is bad and some finite $F_Y \subset Y$ is the evidence. Now the collection $L = \{Y \setminus F_Y \mid Y \in I\}$ is trivially cofinal (in fact $Y \subseteq^* Y \setminus F_Y$). Let $p_1 = (a_0, H_0 \cup \{L\})$ be the extension of p_0 obtained by adding L. Find $q \in D$ that extends p_1. Then, by definition of extension there exists some $A \in L$ such that $a_q \setminus a_0 \subseteq A$ (in fact a cofinal set of such A). Now, $A = Y \setminus F_Y$ for some $Y \in I$, and this is a contradiction.

Thus some $Y \in I$ is not bad, and we can take it to be in M (which is an elementary substructure). Hence $Y \subseteq^* X$. Say $F = Y \setminus X$. Then F is finite, and as Y is not bad there exists some $q \in D$ extending p_0 and such that $a_q \setminus a_0 \subseteq Y \setminus F \subseteq X$. q.e.d.-Lemma

But this lemma alone is not enough and there is an additional problem. To obtain that p_ω is a pre-extension of p_i, we require that

$$\text{for every } K \in H_i: \ \delta(e_i, K) \text{ is cofinal in } I, \tag{1}$$

where $e_i = (\bigcup_{j\in\omega} a_j) \setminus a_i$. For this aim, we shall also define a sequence $X^i_M \in I$, $X^{i+1}_M \subseteq X^i_M$ of almost covers of $I \cap M$, and at the i-th stage we will get p_{i+1} so that

$$a_{i+1} \setminus a_i \subset X^i_M. \tag{2}$$

For this strategy to work, we need another lemma.

Lemma 1.10 *Suppose $X \in I$ and $L \subseteq I$ is cofinal. Then for some finite $F \subset X$, $\delta(X \setminus F, L)$ is cofinal.*

Proof of the lemma. Suppose that this is not true and for every finite $F \subset X$ the set $\delta(X \setminus F, L) = \{A \in L \mid X \setminus F \subseteq A\}$ is not cofinal. So let $Y(F) \in I$ be such that there is no $A \in L$ with $X \setminus F \subseteq A$ that almost covers $Y(F)$. Since I is a P-ideal, there is a set A in I that almost covers X and each of the $Y(F)$ sets for $F \in [X]^{<\omega}$. As L is cofinal in I, we can take $A \in L$. But now $F = X \setminus A$ yields a contradiction. q.e.d.-Lemma.

Now the construction of the M-generic sequence $p_i \in P \cap M$ can be described with more details. We have an enumeration of all dense subsets of P that are in M, and we require in defining P_{j+1} that we enter the jth dense set in this enumeration. But we have (1) as an additional mission. Since every H_i is countable, we can fix an enumeration of H_i, and so for every $K \in H_i$ there is a stage $j \geq i$ so that in defining p_{j+1} (assuming p_j is already defined) we are required to take care of K. Since $p_i \leq p_j$, $L = \delta(a_j \setminus a_i, K) \in H_j$ is one of the cofinal sets there. Applying Lemma 1.10 to $X^j_M \in I$ and L, we get a finite $F \subset X^j_M$ such that $\{A \in L \mid X^j_M \setminus F \subseteq A\}$ is cofinal in I. So if we define $X^{j+1}_M = X^j_M \setminus F$ then (2) ensures that for every $k > j$ $a_k \setminus a_j \subseteq X^{j+1}_M$, and so $e_j = (\bigcup_{k>j} a_k) \setminus a_j \subseteq X^{j+1}_M$. Hence $\delta(e_j, L) \supseteq \delta(X^{j+1}_M, L)$, and so $\delta(e_j, L)$ is cofinal in I. But $\delta(e_i, K) = \delta(e_j, L)$ and hence $\delta(e_i, K)$ will be cofinal.

PID with CH

Now to get the consistency with CH of the dichotomy principle for P ideals, we make first the following definition. Say that $X \in [\omega_1]^{\leq\omega}$ is a good cover of M iff the following hold (we do not ask $X \in I$):

1. For every $A \in I \cap M$, $A \subseteq^* X$.
2. There is a function G_X which assigns to every cofinal $K \subseteq I$ in M a finite set $F = G_X(K) \subset X$ such that $\{A \in K \mid X \setminus F \subseteq A\}$ is cofinal.

Observe that in fact we can deal with countably many good covers of M (where M is now not an elementary substructure but just a transitive model or an isomorphism type). For every pair (X, G_X) we construct a corresponding generic sequence. We also observe that P is in fact α-proper for every countable α. So the machinery of Dee-complete forcing can be applied.

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