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INDEPENDENCE RESULTS

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Abstract. We prove independence results concerning the number of nonisomorphic models (using the S-chain condition and S-properness) and the consistency of "ZCF + $2^{\aleph_0} = \aleph_2$ + there is a universal linear order of power $\aleph_1$". Most of these results were announced in [Sh 4], [Sh 5].

In subsequent papers we shall prove an analog f MA for forcing which does not destroy stationary subsets of ω_1, investigate $\mathscr{D}$-properness for various filters and prove the consistency with G.C.H. of an axiom implying SH (for $\aleph_1$), and connected results.

§1. Isomorphism of subsets of $^{\omega}2$. In this section, for sequences η, ν, $\eta E_n^{\mathrm{sq}} \nu$ iff $\eta \restriction n = \nu \restriction n$, so E_n^{sq} is an equivalence relation.

By [Sh 1, Theorem VIII, 1.8] if T is a complete $\aleph_0$-unstable (first-order) theory, $T \subseteq T_1$, $|T_1| = \aleph_0$, then for $\aleph_0 < \lambda \le 2^{\aleph_0}$, $I(\lambda, T_1, T) = 2^\lambda (I(\lambda, T_1, T)$, the number of nonisomorphic models of cardinality λ, which are in $\mathrm{PC}(T_1, T)$ = the class of $L(T)$-reducts of models of T_1). By [Sh 1] we can replace $|T_1| = \aleph_0$ by $|T_1| \le \lambda$, $2^{\aleph_0} < 2^\lambda$ (even less). It was also proved in [Sh 1, Theorem VIII, 2.20] that for any complete $\aleph_0$-unstable T, $\aleph_0 < \lambda \le 2^{\aleph_0}$, $|T| \le \lambda$, $I(\lambda, T) = 2^\lambda$. It was asked (see [Sh 1, VIII]) whether for any complete $\aleph_0$-unstable T, $T \subseteq T_1$, $|T_1| \le \lambda < 2^{\aleph_0}$, $I(\lambda, T_1, T) = 2^\lambda$, i.e. whether we can prove this in ZFC. We shall show that this is not the case.

THEOREM 1.1. (1) *Let* $T = \mathrm{Th}((^{\omega}2, E_0^{\mathrm{sq}}, E_1^{\mathrm{sq}}, \ldots))$ (*so* T *is countable complete*, $\aleph_0$-*unstable but superstable*). *Then, if ZFC is consistent, it is consistent with ZFC, that for some* T_1, $T \subseteq T_1$, $|T_1| = \aleph_1$, $I(\aleph_1, T_1, T) = 1$. *Also* T *has a universal model* in $\aleph_1$.

Notation 1.2. Let T_1 be the following theory: it consists of (i) T, (ii)$_n$ a sentence saying that for any x_0, y_0, $n < \omega$, $z \mapsto F_n(x_0, y_0, z)$ is an automorphism of the $L(T)$-reduction of the model taking x_0/E_n to y_0/E_n, and (iii) $z \mapsto F(x_0, z)$ is a one-to-one function from the model into $\{x\colon x E_n x_0$ for all $n < \omega\}$, (iv) $\{E_n(c_\eta, c_\nu)^{\mathrm{if}(\eta \restriction n = \nu \restriction n)}\colon \eta, \nu \in {}^{\omega}2, \eta, \nu$ are constructible (i.e. in $L\}$ (c_η is an individual constant) where $\varphi^{\mathrm{if}\,(\mathrm{statement})}$ is φ if the statement is true and $\neg\varphi$ otherwise. Let $\eta \prec \nu$ mean η is an initial segment of ν.

Claim 1.3. (1) T_1 is a theory (i.e. is consistent), $T \subseteq T_1$, $|T_1| = \aleph_1$ and every model in $\mathrm{PC}(T_1, T)$ of cardinality $\aleph_1$ is isomorphic to a model of the form $(A \times \omega_1, E_0, E_1, \ldots)$ where

(i) $A \subseteq {}^{\omega}2$, $(\eta, \alpha) E_n (\nu, \beta)$ iff $\eta E_n^{\mathrm{sq}} \nu$,

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(ii) for every $\eta, \nu \in {}^\omega 2$ there is an automorphism of $(A \cup {}^{\omega>}2, \prec)$ taking η to ν,

(iii) A includes all the constructible members of ${}^\omega 2$.

Claim 1.4. For proving 1.1, it suffices to prove the following. Suppose (for simplicity only), $V = L$, and find a forcing notion P, such that:

(1) P satisfies the countable chain condition, $|P| = \aleph_2$ hence in V^P, $2^{\aleph_0} = \aleph_2$;

(2) in V^P, R^L (the set of constructible reals) is of the second category;

(3) if $A, B \subseteq {}^\omega 2$ are everywhere of the second category, $|A| = |B| = \aleph_1$, then $(A \cup {}^{\omega>}2, \prec) \cong (B \cup {}^{\omega>}2, \prec)$.

Let us first concentrate on the induction step.

Claim 1.5. Suppose for every limit $\delta < \omega_1$, $\mathscr{S}_\delta$ is a countable family of subsets of δ.

Suppose W is a subset of ${}^{\omega>}\omega$ closed under initial segments, and $W_\omega = \{\eta \in {}^\omega\omega: n < \omega \to \eta \restriction n \in W\}$. So on W_ω a natural topology is defined ($\{\eta \in W_\omega: \nu \prec \eta\}$ for any $\nu \in W$ is a basic open set). Let P^0 be a countable set of partial automorphisms [i.e. functions f, with domain and range $\subseteq {}^\omega 2$, preserving $\prec$ and length] of $(W, \prec)$ such that

(a) for $f \in P^0$, Dom f is closed under initial segments;

(b) if $f \in P_0$, g a finite automorphism, $f \cup g$ a partial automorphism then $f \cup g \in P^0$.

Suppose A, B are subsets of W_ω everywhere of the second category, and $f \in P^0$, $\eta \in A, \nu \in B$ implies $\{\eta \restriction n : n < \omega\} \cap \mathrm{Dom} f$, $\{\nu \restriction n: n < \omega\} \cap \mathrm{Range} f$ are finite; *then* there is a forcing notion P, such that

(A) $|P| = \aleph_1$;

(B) $\emptyset \Vdash^P$ "$(A \cup W, \prec), (B \cup W, \prec)$ are isomorphic";

(C) there is a one-to-one function f from P onto ω_1 such that for any limit $\delta < \omega_1$, and $S \in \mathscr{S}_\delta$, $f^{-1}(S) = \{p \in P: f(p) \in S\}$ is predense in P, provided that it is predense in $f^{-}(\delta)$;

(D) $P^0 \subseteq P$, moreover we can assume f extends a given one-to-one function f_0 from P_0 onto ω.

PROOF OF 1.5. Let P_0 be the set of partial isomorphisms f from $(A \cup W, \prec)$ into $(B \cup W, \prec)$ such that for some $g \in P^0$, and $\eta_0, \ldots, \eta_{k-1} \in A$, $g \subseteq f$ and

$$\mathrm{Dom} f - \mathrm{Dom}\, g \subseteq \{\eta_l \restriction \alpha: l < k, \alpha \le \omega\}.$$

So f determines $\eta_0, \ldots, \eta_{k-1}$ uniquely. Let $m(f) = \min\{m < \omega$: for $l < k$, $\eta_l \restriction m \prec \nu \in \mathrm{Dom} f$ implies $\nu \prec \eta_l\}$ and let $f \restriction m = f \restriction \{\eta \in W$: for no l, $\eta \prec \eta_l$, *or* $l(\eta) \le m\}$ (so $f \restriction m \in P^0$ when $m \ge m(f)$).

P_0 satisfies condition (B) but not necessarily (C) (notice (C) will assure us usually that P satisfies the countable chain condition, which does not hold here, as the number of possible images of any $\eta \in A$ is $\aleph_1$).

We shall define by induction on $i < \omega_1, f_i, A_i, B_i$ $(i < \omega_1)$ such that:

(a) $A_i \subseteq A$, $B_i \subseteq B$ and both are countable;

(b) the A_i are pairwise disjoint, similarly the B_i;

(c) A_i, B_i are dense (i.e., not disjoint to any basic subset of $A \cup W$);

(d) let $A = \{x_i^0: i < \omega_1\}$, $B = \{x_i^1: i < \omega_1\}$, then $x_\alpha^0 \in \bigcup_{i \le 2\alpha} A_i$, $x_\alpha^1 \in \bigcup_{i \le 2\alpha+1} B_i$;

(e) let $\bar{A}^\alpha = \langle A_i: i < \alpha\rangle$, $\bar{B}^\alpha = \langle B_i: i < \alpha\rangle$, and

$$P^\alpha = P(\bar{A}^\alpha, \bar{B}^\alpha) = \{g \in P_0: \mathrm{Dom}\, g \subseteq \bigcup_{i<\alpha} A_i \cup W, x \in A_j \Leftrightarrow g(x) \in B_j\},$$

Then f_α is a one-to-one function from P^α onto $\omega(1+\alpha)$;

(f) f_i is increasing;

(g) if $\beta \leq \alpha$, $S \in \mathscr{S}_{\omega(1+\beta)}$, $f_\beta^{-1}(S)$ is predense in P^α, $f \in P^\alpha$, $g \in P^0$ *then* either for some $h \in f_\beta^{-1}(S), f \cup g \cup h \in P^\alpha$ or for some $m < \omega$ for no $h \in f_\beta^{-1}(S)$, $(f \upharpoonright m) \cup g \cup h \in P^\alpha$.

Clearly it suffices to assume A_i, B_i $(i < \alpha)$, f_α are defined, and are as required, and to define appropriately A_α, B_α, $f_{\alpha+1}$. For this let $W = \{\eta_l : l < \omega\}$ and we shall define by induction on $k < \omega$, sets A^k, B^k such that:

(i) $A^k \subseteq A - \bigcup_{i<\alpha} A_i$, $B^k \subseteq B - \bigcup_{i<\alpha} B_i$, A^k $(k < \omega)$ is increasing and B^k $(k < \omega)$ is increasing;

(ii) $|A^k| + |B^k| \leq k + 1$;

(iii) if α is even, let $j < \omega_1$ be minimal such that $x_j^0 \notin \bigcup_{i<\alpha} A_i$, and let $A^0 = \{x_j^0\}$, $B^0 = \emptyset$, and if α is odd let j be minimal such that $x_j^1 \notin \bigcup_{i<\alpha} A_i$ and let $A^0 = \emptyset$, $B^0 = \{x_j^1\}$;

(iv) if $k = 2l + 1$ for some $\eta \in A^k$, $\eta_l \prec \eta$, and if $k = 2l + 2$ for some $\eta \in B^k$, $\eta_l \prec \eta$;

(v) $P = P(\bar{A}^{\alpha\frown}\langle A^k\rangle, \bar{B}^{\beta\frown}\langle B^k\rangle)$ (defined as in (e)) satisfies (g).

If we succeed we shall let $A_\alpha = \bigcup_{k<\omega} A^k$, $B_\alpha = \bigcup_{k<\omega} B^k$. By (ii), A_α, B_α are countable; by (i) they are disjoint to $\bigcup_{i<\alpha} A_i$, $\bigcup_{i<\alpha} B_i$ resp.; by (iv) they are dense; by (iii) they satisfy (d). Last and most important, (g) is satisfied because

$$P^{\alpha+1} = \bigcup_{k<\omega} P(\bar{A}^{\alpha\frown}\langle A^k\rangle, \bar{B}^{\beta\frown}\langle B^k\rangle)$$

by the finite character of P_0 (and as being compatible in $P^{\alpha+1}$ is equivalent to being compatible in P_0).

Then we have to define $f_{\alpha+1}$ which presents no problems. So let us carry the induction on k.

Case 1. $k = 0$.

Define A^0, B^0 by (iii). Thus (i), (ii) hold trivially, (iv) is empty, and (v) immediate as by definition

$$P(\bar{A}^{\alpha\frown}\langle A^0\rangle, \bar{B}^{\alpha\frown}\langle B^0\rangle) = P(\bar{A}^\alpha, \bar{B}^\alpha) = P^\alpha$$

and P^α satisfies (g).

Case 2. $k = 2l + 1$ and we want to define for $k + 1$.

Same as Case 3.

Case 3. $k = 2l$, and we want to define for $k + 1$.

If we can find $\eta \in C_0 = \{\eta : \eta \in A - \bigcup_{i<\alpha} A_i \cup A^k, \eta_l \prec \eta\}$ such that $A^{k+1} = A^k \cup \{\eta\}$, $B^{k+1} = B^k$ satisfy (v) then (as (i)–(iv) hold trivially) we are finished. So for every $\eta \in C_0$ there are: $\beta = \beta_\eta$, $S_\eta \in \mathscr{S}_{\omega(1+\beta)}$ such that $f_\alpha^{-1}(S_\eta)$ is predense in P^α (hence in $P(\bar{A}^{\alpha\frown}\langle A^k\rangle, \bar{B}^{\alpha\frown}\langle B^k\rangle)$, and $g_\eta \in P^0$, and $f_\eta \in P(\bar{A}^{\alpha\frown}\langle A^k\rangle, \bar{B}^{\alpha\frown}\langle A^k\rangle)$ and $\nu_\eta \in B^k$, such that: f'_η, the extension of f_η by $\eta \upharpoonright \alpha \mapsto \nu_\eta \upharpoonright \alpha$ $(\alpha \leq \omega)$ belong to P_0, is compatible with g_η, but not with $g_\eta \cup h$ for any $h \in f_\alpha^{-1}(S_\eta)$, but for every $m < \omega$, for some $h \in f_\alpha^{-1}(S_\eta)$, $f'_n \upharpoonright m \cup g_\eta \cup h \in P_0$.

As A is everywhere of the second category, and $\bigcup_{i<\alpha} A_i \cup A^k$ is countable clearly C_0 is of the second category. The number of possible S_η is countable (as $\bigcup_{\beta\leq\alpha} S_{\omega(1+\beta)}$ is), and similarly f_η, ν_η, g_η. Hence for some $C \subseteq C_0$ of the second category, $S \in$

$S_{\omega(1+\beta)}$, $f_0 \in P(\bar{A}^{\alpha\frown}\langle A^k\rangle, \bar{B}^{\alpha\frown}\langle B^k\rangle)$, $g_0 \in P^0$, $\nu_0 \in B^k$ for every $\eta \in C$, $S_\eta = S, f_\eta = f_0$, $\nu_\eta = \nu_0$, and $g_\eta = g_0$.

We shall prove that C is nowhere dense, hence get a contradiction so finishing the induction on k and thus finishing the induction on α, and this will clearly finish the proof of 1.5; for (B) note the sets $D_{\eta,\nu} = \{f\colon \eta \in \operatorname{Dom} f,\ \nu \in \operatorname{Range} f\}$ are predense.

So suppose $\rho \in W$, and we shall find $\rho' \in W$, $\rho \prec \rho'$ such that $\{\eta \in W_\omega\colon \rho' \prec \eta\}$ is disjoint to C. If $\rho' = \rho$ is not as required, choose $\eta \in C$, $\rho \prec \eta$, let $f_1 = f_0 \cup \{\langle \eta \restriction \gamma, \nu_0 \restriction \gamma\rangle\colon \gamma \le \omega\}$, so for every $m \ge m(f_1)$ for some $h_m \in f^{-\frac{1}{2}}(S)$ $((f_1 \restriction m) \cup g_0 \cup h_m) \in P_0$. Let $l = l(\rho) + m(f_1) + 1$ and $m > l$ minimal such that $\nu_0 \restriction m \notin \operatorname{Range}((f_1 \restriction l) \cup g_0 \cup h_l)$ and ρ^1 be that $((f_1 \restriction l) \cup g_0 \cup h_l)(\rho^1 \restriction (m-1)) = \nu_0 \restriction (m-1)$, and $\rho^1 \notin \operatorname{Dom}((f_1 \restriction l) \cup g_0 \cup h_l)$. Clearly $\rho^1 \prec \eta_0 \in W_\omega \cap C$ implies

$$(f_0 \cup \{\langle \eta^0 \restriction \gamma, \nu_0 \restriction \gamma\rangle\colon \gamma \le \omega\}) \cup g_0 \cup h_l \in P_0.$$

So $\rho' \prec \eta^1 \in W_\omega \to \eta^1 \notin C$.

Using the lemma we can choose $\mathscr{S}_\delta$ appropriately, assure P satisfy the countable chain condition and R^L remains of the second category (see below). However, if we repeat the iteration as in Solovay and Tenenbaum [ST], it is not clear why R^L remains of the second category. So we introduce:

Definition 1.6. (1) A sequence $\mathbf{S} = \langle \mathscr{S}_\delta\colon \delta < \omega_1 \text{ limit}\rangle$ is an $\aleph_1$-oracle if

(i) $\mathscr{S}_\delta$ is a transitive countable set, $\alpha + 1 \subseteq \mathscr{S}_\delta$, and $\mathscr{S}_\delta$ is quite closed, e.g. it is a model of ZFC except the power-set axiom;

(ii) for any $A \subseteq \omega_1$, $\{\delta < \omega_1\colon A \cap \delta \in \mathscr{S}_\delta\}$ is stationary.

(2) $\mathbf{S}'$ is a proper extension of $\mathbf{S}$ if: $\mathscr{S}_\alpha \in \mathscr{S}'_\alpha$ and there is $f \in \mathscr{S}'_\alpha$ from ω onto $\mathscr{S}_\alpha$, for a closed unbounded set of α's.

(3) We call $\mathbf{S}$ a strong $\aleph_1$-oracle if in (ii) the set contains a closed unbounded set.

Claim 1.7. (1) If $\diamondsuit_{\aleph_1}$ holds, then there is an $\aleph_1$-oracle.

(2) Every $\aleph_1$-oracle has a proper extension; in fact every $\aleph_1$ $\aleph_1$-oracles has a common proper extension.

(3) If $\mathbf{S}$ is an $\aleph_1$-oracle, M_α a model with universe ω_1 and countable language then $\{\alpha\colon \langle M_i \restriction \alpha\colon i < \alpha\rangle \in \mathscr{S}_\alpha\}$ is stationary.

Definition 1.8. A forcing notion P satisfies the $\mathbf{S}$-chain condition ($\mathbf{S}$ an $\aleph_1$-oracle), if for every $P' \subseteq P$, $|P'| \le \aleph_1$, there are P'', $P' \subseteq P''$, $|P''| \le \aleph_1$, and a one-to-one function f from P'' into ω_1, and $A \subseteq \omega_1$, such that:

(*) if $A \cap \delta \in \mathscr{S}_\delta$ (δ limit $< \omega_1$), $S \in \mathscr{S}_\delta$, $S \subseteq \delta$, and $f^{-1}(S)$ is predense in $f^{-1}(\delta) = \{p \in P''\colon f(p) < \delta\}$ then $f^{-1}(S)$ is predense in P.

Remark. Notice that for an $\aleph_1$-oracle $\mathbf{S}$, $\mathscr{D}(\mathbf{S}) = \{\{\delta\colon A \cap \delta \in \mathscr{S}_\delta, \delta < \omega_1\}\colon A \subseteq \omega_1\}$ is a nomal filter over ω_1, so our demand is: "there is $B \in \mathscr{D}(\mathbf{S})$ such that for every $\delta \in B\ldots$".

Claim 1.9. If P satisfies the $\mathbf{S}$-chain condition then

(1) for any one-to-one function f from P'' into ω_1, for some $A \subseteq \omega_1$, (*) of Definition 1.7 holds;

(2) P satisfies the $\aleph_1$-chain condition.

Lemma 1.10. *If* $\mathbf{S}$ *is a proper extension of some* $\mathbf{S}'$ (*both* $\aleph_1$-*oracles*), *and* P *satisfy the* $\mathbf{S}$-*chain condition, then in* V^P, R^V *is still of the second category* (*similarly for any* $(W_\omega)^V$ *for* W *as in* 1.5).

PROOF. Suppose not. Thus in V^P there is $A = \bigcup A_n \supseteq R^V$, A_n closed nowhere dense sets of reals. So $A = \langle A_n : n < \omega \rangle$ has a name $\underset{\sim}{A} = \langle \underset{\sim}{A}_n : n < \omega \rangle$ and there are $P^k_{\nu,n}$, $\rho^k_{\nu,n}$ $(k < \omega, \nu \in {}^{\omega>}2, n < \omega)$ such that:

(a) $\{p^k_{\nu,n} : k < \omega\}$ is a maximal antichain in P;

(b) $p^k_{\nu,n} \Vdash^P$ "$\underset{\sim}{A}_n$ is disjoint to $\{\eta \in {}^{\omega}2 : \rho^k_{\nu,n} \prec \eta\}$";

(c) $\nu \prec \rho^k_{\nu,n} \in {}^{\omega>}2$.

Apply Definition 1.7 for $P' = \{p^k_{\nu,n} : k, n < \omega, \nu \in {}^{\omega>}2\}$ so we have appropriate P'', $P' \subseteq P'' \subseteq P$, $|P''| \leq \aleph_1$, $f: P'' \to \omega_1$, and $A \subseteq \omega_1$. As S′ is an $\aleph_1$-oracle and, S a proper extension, for some limit $\delta < \omega_1$, $A \cap \delta \in \mathscr{S}'_\delta$, and $\{\langle k, n, \nu, \rho^k_{\nu,n}\rangle : k, n < \omega, \nu \in {}^{\omega>}2\} \in \mathscr{S}'_\delta$ and $\{(f(p), f(q)) : p, q \in P'', p \leq q\} \in \mathscr{S}'_\delta$ and $\mathscr{S}'_\delta \in \mathscr{S}_\delta$.

Every closed nowhere dense subset of ${}^{\omega}2$ is representable by a real, so there are only countable many such sets representable in $\mathscr{S}'_\delta$, so as $\mathscr{S}_\delta$ "consider" $\mathscr{S}'_\delta$ as a countable set there is $\eta^* \in \mathscr{S}^*_\delta$ which is in no nowhere dense subsets of ${}^{\omega}2$ representable in $\mathscr{S}'_\delta$. Let

$$Q_n = \{p^k_{\nu,n} : k < \omega, \rho^k_{\nu,n} \prec \eta^*, \nu \in W\}.$$

So clearly $f(Q_n) \in \mathscr{S}_\delta$: If it is predense in P, then $\phi \Vdash^P$ "$\eta^* \notin \underset{\sim}{A}_n$" for every n and it is predense in $f^{-1}(\delta)$ by the choice of η^*. So we finish by the S-chain condition.

LEMMA 1.11. *If P_α $(\alpha < \delta)$ is an increasing continuous sequence of forcing notions satisfying the* S*-chain condition, then their direct limit satisfies it too.*

LEMMA 1.12. *Suppose P satisfies the* S*-chain condition, $|P| \leq \aleph_1$. Then in V^P there is an $\aleph_1$-oracle* S′ *(defined uniformly) such that if in V^P Q satisfies the* S′*-chain condition, then $P * Q$ satisfies the* S*-chain condition in V.*

PROOF. W.l.o.g. the set of elements of P is ω_1, such that for limit δ, if $p, q < \delta$ have a common upper bound, then they have a common upper bound $< \delta$. Let $U = \{\delta : \delta < \omega_1$, limit, and every $S \in \mathscr{S}_\delta$ which is predense in $P \restriction \delta$ is predense in $P\}$.

We define $\mathscr{S}'_\delta$ as $\mathscr{S}_\delta \cup \{a \in V^P : a$, and every member of its transitive closure, have a name (in V) which is in $\mathscr{S}_\delta\}$ if $\delta \in U$, and as $\mathscr{S}_\delta$ otherwise. We leave the details to the reader.

PROOF OF THEOREM 1.1, CLAIM 1.4. Just combine all claims and lemmas. Note that by Claim 1.5 for a given $\aleph_1$-oracle S we can construct P as there with the S-chain condition.

REMARK. Noting 1.7(2), clearly instead of working with an S, iterating $\aleph_2$ times, we can at each stage add one more S. Thus we can make 1.4 true not only for ${}^{\omega>}2$, but for any W as in 1.5.

REMARK. It may be instructive here to remember Baumgartner [B]. He proved the consistency of ZFC + $2^{\aleph_0} = \aleph_2$ with "every two subsets of R (reals, in the usual sense) of cardinality $\aleph_1$ which are $\aleph_1$-dense (i.e. any interval has cardinality $\aleph_1$) are isomorphic". He makes iterated forcing satisfying the countable chain condition. Given two such sets of reals, A, B the obvious forcing (finite isomorphism), fail to satisfy the countable chain condition so he decomposes each to $\aleph_1$ pairwise disjoint countable dense sets A_α $(\alpha < \omega_1)$, B_α $(\alpha < \omega_1)$ and looks only at finite isomorphisms f, $x \in A_\alpha = f(x) \in B_\alpha$. Now the obvious reason for the failure of the countable chain condition disappears, and using CH he proves we can find a decomposition in which it holds. But as he iterates ω_2 times, CH is still true in any intermediate step, so he can finish.

We use similar decompositions, but in their construction we use the fact that A, B are of the second category and we have to preserve the fact that $({}^{\omega}2)^L$ remains of the second category. In particular, it is not obvious why this will be preserved at limit stages. So we use the method of [Sh 2], represented differently, i.e. the S-chain condition. Note that we could have alternatively used forcing as in the proof of the consistency of the nonexistence of P-points (see Wimmers [W]).

However, Baumgartner's forcing P cannot satisfy an S-chain condition. This will be discussed somewhere else. For this, and a more detailed exposition of forcing with the oracle chain condition, see [Sh 3].

§2. Nonsuperstable theories. By [Sh 1, VII, §2], if T is not superstable, $T \subseteq T_1$, $\lambda \geq |T_1| + \aleph_1$ then $I(\lambda, T_1, T) = 2^\lambda$, except maybe when all the following conditions hold: $\lambda = |T_1|$, $T_1 \neq T$, $\lambda^{\aleph_0} > \lambda$, and even $\lambda < \sum_{\mu<\lambda}\mu^{\aleph_0}$. It was asked whether we can omit those restrictions, and we shall show that we cannot.

THEOREM 2.1. *Let* $T = \mathrm{TH}({}^{\omega}(\omega_1)E_0^{\mathrm{sq}}, E_1^{\mathrm{sq}}, \ldots)$ *(so T is a countable, complete, stable but not superstable theory). Then (if ZFC is consistent) it is consistent with ZFC that for some* T_1, $T \subseteq T_1$, $|T_1| = \aleph_1$ *and* $I(\aleph_1, T_1, T) = 1$. *Also T has a universal model of power* $\aleph_1$.

Notation 2.2. Let T_1 be the following theory: it consists of T, for $n < \omega$ a sentence saying that for any x_0, y_0, $z \mapsto F_n(x_0, y_0, z)$ is a mapping from x_0/E_n onto y_0/E_n, preserving all the E_m's and $z \mapsto F(x_0, z)$ is a one-to-one function from the model into $\{x: xE_nx_0$ for all $n < \omega\}$, and $\{E_n(c_\eta, c_\nu)^{\mathrm{if}(\eta\restriction n=\nu\restriction n)}: \eta, \nu \in {}^{\omega}2$ are constructible$\}$ (c_η an individual constant).

Claim 2.3. (1) T_1 is a theory (i.e. consistent) and every model in $\mathrm{PC}(T_1, T)$ of cardinality $\aleph_1$ is isomorphic to a model of the form $(A \times \omega_1, E_0, E_1, \ldots)$ where

(i) $A \subseteq {}^{\omega}\omega_1$;

(ii) $(\eta, i)E_n(\nu, j)$ iff $\eta \restriction n = \nu \restriction n$.

Claim 2.4. For proving 2.1, it suffices to prove the following.

Suppose (for simplicity only) that $V = L$, and find a forcing notion P such that

(1) $|P| = \aleph_2$, P does not collapse $\aleph_1$, $\aleph_2$ and in V^P, $2^{\aleph_0} = \aleph_2$;

(2) in V^P, $({}^{\omega}\omega_1) \cap L$ is a subset of ${}^{\omega}\omega_1$ of the second category;

(3) if ${}^{\omega}\omega_1 \cap L \subseteq A, B \subseteq {}^{\omega}\omega_1$, and for each limit $\delta < \omega_1$, $\eta^* \in {}^{\omega>}\delta$, $A \cap \{\eta \in {}^{\omega}\delta: \eta \notin \bigcup_{\alpha<\delta}{}^{\omega}\alpha, \eta^* \prec \eta\}$, $B \cap \{\eta \in {}^{\omega}\delta: \eta \notin \bigcup_{\alpha<\delta}{}^{\omega}\alpha, \eta^* \prec \eta\}$ are of the second category, then $(A \cup {}^{\omega>}\omega_1, \prec) \cong (B \cup {}^{\omega>}\omega_1, \prec)$.

§3. Proper forcing.

DEFINITION 3.1. (1) A forcing notion P is λ-proper, where λ is a regular cardinal (if $\lambda = \aleph_1$ we omit it) provided that the following holds.

If $I_\alpha = \{p_i^\alpha: i < i_\alpha\}$ is predense in P for $\alpha < \alpha_0$, $i^* = \bigcup_{\alpha<\alpha_0} i_\alpha \cup \alpha_0$, $p \in P$ *then* $\mathbf{U} = \{U: U \subseteq i^*, |U| < \lambda, U \cap \lambda$ is an initial segment of λ and there is a $q_U \in P$, $q_U \geq p$, above which $I_\beta^U = \{P_i^\beta: i \in U\}$ is predense, for each $\beta \in U\}$ contains a closed unbounded subset of $\mathbf{P}_{<\lambda}(i^*) = \{U: U \subseteq i^*, |U| < \lambda\}$.

(2) $\mathbf{U}$ is a closed unbounded subset of $\mathbf{P}_{<\lambda}(i^*)$ if for some model M with countable language with universe i^*, $\mathbf{U} = \{|N|: N \prec M, \|N\| < \lambda, |N| \cap \lambda$ an initial segment of $\lambda\}$. The last phrase is not needed for $\lambda = \aleph_1$. An alternative definition is: $\mathbf{U}$ is

unbounded $[\forall A \in \mathbf{P}_{<\lambda}(i^*)][\exists B \in \mathbf{P}_{<\lambda}(i^*)](A \subseteq B)$ and closed under union of increasing sequences of length λ.

(3) In (2) we shall say M exemplifies $\mathbf{U}$ is closed unbounded (or for (1) contains a closed unbounded set).

Claim 3.2. (1) In Definition 3.1 it suffices to assume it holds for I_α maximal antichains.

(2) If Definition 3.1 holds, then for any ordinal α_0, $p \in P$ and predense subsets $I_t = \{p^t_i: i \in A_t \subseteq \alpha_0\}$ of P, $(t \in A \subseteq {}^{\omega>}\alpha_0)$ for a closed unbounded set of $U \in \mathbf{P}_{<\lambda}(\alpha_0)$, for some $p_U \in \mathbf{P}$, $I^U_t = \{p^t_i: i \in A_t \cap U\}$ is predense above p_U for every $t \in ({}^{\omega>}U) \cap A$, and $p_U \geq p$.

PROOF. (1) We assume the statement in Definition 3.1 holds for antichains, suppose $I_\alpha = \{p^\alpha_i: i < i_\alpha\}$ $(\alpha < \alpha_0)$ are predense in P, and show $\mathbf{U}$ contains a closed unbounded subset of $\mathbf{P}_{<\lambda}(i^*)$. (The other direction is trivial.)

For each α, among the antichains $\{q_j: j < j'\} \subseteq P$ satisfying: for each $j < j'$ for some $i < i_\alpha, p^\alpha_i \leq q_j$, choose a maximal one $J_\alpha = \{q^\alpha_j: j < j_\alpha\}$ and let $f_\alpha: j_\alpha \to i_\alpha$ be such that $p^\alpha_{f_\alpha(j)} \leq q^\alpha_j$. Clearly J_α exist by Zorn's Lemma. It is maximal (among antichains in general) as I_α is predense.

By the hypothesis there is a model M_0 with countable language, such that for any $N \prec M_0$, $\|N\| < \lambda$, $|N| \cap \lambda$ an ordinal there is $r_N \geq p$, $r_N \in P$, above which $J^N_\alpha = \{q^\alpha_j: j < j_\alpha, j \in N\}$ is predense for each $\alpha \in N$. Let f be a two-place function on α_0, $f(\alpha, j) = f_\alpha(j)$ when defined and zero otherwise. $M = (M_0, f)$ is a model exemplifying the properness of P for the I_α.

(2) Easy.

LEMMA 3.3. *Suppose λ is regular, P is λ-proper (in V), and Q is λ-proper in V^P. Then $P * Q$ is λ-proper in V.*

PROOF. Remember that elements of $P * Q$, are pairs $(p, \mathbf{q})$, $\phi \Vdash^P$ "$\mathbf{q} \in \mathbf{Q}$", and $(p_1, \underset{\sim}{q}_1) \leq (p_2, \underset{\sim}{q}_2)$ iff $P \models p_1 \leq p_2$ and $p_2 \models^P$ "$\underset{\sim}{q}_1 \leq \underset{\sim}{q}_2$". So let $I_\alpha = \{(p^\alpha_i, \underset{\sim}{q}^\alpha_i): i < i_\alpha\}$ be maximal antichains in $P * Q$ for $\alpha < \alpha_0$. Let G be a generic set for P, and $A_\alpha = \{i < i_\alpha: p^\alpha_i \in G\}$, $J^0_\alpha = \{q^\alpha_i: i \in A_\alpha\}$. As Q is λ-proper, there is a model M_1 as in Definition 3.1, and w.l.o.g. it has Skolem functions $F_n(\bar{x}_n)$. Let $\mathbf{M}_1$, $\mathbf{F}_n(\bar{x}_n)$ be their names in V and for notational simplicity let $i^* \geq |P|$. So for every $\bar{a} \in |\mathbf{M}_1| = i^*$ there is a maximal antichain $J^n_{\bar{a}} = \{p^{\bar{a},n}_i: i < i_{\bar{a},n} \leq i^*\} \subseteq P$, and a function f such that $p^{\bar{a},n}_i \Vdash^P$ "$\mathbf{F}_n(\bar{a}) = f(\bar{a}, n, i)$" (and $f(\bar{a}, n, i)$ is an ordinal $< i^*$).

Now we apply Definition 3.1 (or more exactly Claim 3.2(2)) to the family:

$$I'_\alpha = \{p^\alpha_i: i < i_\alpha\}, \qquad J^n_{\bar{a}}\,(\alpha < \alpha_0, \bar{a} \in {}^{\omega>}(i^*), n < \omega)$$

and get a model M_0, $|M_0| = i^*$. Let $M = (M_0, f)$ and we shall prove it exemplifies the demand from Definition 3.1 for the I_α.

So let $N \prec (M, f)$, $\|N\| < \lambda$, $|N| \cap \lambda$ be an ordinal. By its definition, there is $p_N \in P$, $p_N \geq p$, such that above it

$$\{p^\alpha_i: i < i_\alpha, i \in N\}, \qquad \{p^{\bar{a},n}_i: i < i_{\bar{a},n}, i \in N\}$$

are predense, for $\alpha \in N$, $\bar{a} \in N$. By the definition of $J^n_{\bar{a}}$, and as $|N|$ is closed under f, p_N force $F_n(\bar{a})$ to be in $|N|$. Let $G \subseteq P$ be generic, $p_N \in G$. As M_1 (in V^P) has Skolem functions, it forces $|N|$ to be the universe of an elementary submodel of M_1. So in $V^P = V[G]$ there is $q_N \in Q$ such that above it $J^0 = \{q^\alpha_i: p^\alpha_i \in G, i \in |N|\}$

is predense for $\alpha \in N$ and $q_N \geq q$. It is now easy to finish.

LEMMA 3.4. *Suppose $P_{n+1} = P_n * Q_n$, P_0 trivial, and Q_{n+1} is proper in V^{P_n}. Then the inverse limit P of the P_n is proper.*

PROOF. Clearly an element of P has the form $\bar{q} = \langle q_n : n < \omega \rangle$, $q \restriction k \Vdash^{P_{k+1}}$ "$q_k \in \underset{\sim}{Q}_k$". So suppose $I_\alpha = \{\bar{q}^\alpha_i : i < i_\alpha\} \subseteq P$ $(\alpha < \alpha_0)$ are predense in P. For regular $\bar{\kappa}$ big enough let $M_0 = (H(\bar{\kappa}), \in, P, \Vdash)$ ("big enough" means $P \in H(\bar{\kappa})$, and names of elements which will be in $H(\bar{\kappa})$ are in $H(\bar{\kappa})$).

Let $N \prec M_0$, $\|N\| < \lambda$, $|N| \cap \lambda$ be an ordinal, $\bar{p} \in P \cap N$ and we shall prove that there is $\bar{q} \in P$ above which $I_\alpha \cap N$ $(\alpha \in N)$ are predense. This is sufficient, for first expand M_0 to M_1 by adding Skolem functions, then expand M_1 to M_2 by adding for each formula a relation equivalent to it, reduce M_2 to M_3 by throwing away the functions, and M will be the submodel of M_3 with universe $i^* = \bigcup_{\alpha<\alpha_0} i_\alpha \cup \alpha_0$.

Let the ordinals in N be $\{\gamma(l) : l < \omega\}$.

We now define by induction on $n < \omega$, $\underset{\sim}{r}_n \in \underset{\sim}{Q}_n$ (i.e., $\phi \Vdash^{P_n}$ "$\underset{\sim}{r}_n \in \underset{\sim}{Q}_n$") $r^*_n \in P \cap N$ and P_n-names $\underset{\sim}{\sigma}_n$ such that

(a) above $\langle \underset{\sim}{r}_0, \ldots, \underset{\sim}{r}_{n-1} \rangle \in P_n$, $\{\bar{q}^\alpha_i \restriction n : i < i_\alpha, i \in N\}$ is predense, for each $\alpha \in N$;

(b) $r^*_n \leq r^*_{n+1}$, $r^*_n \restriction n \leq \langle \underset{\sim}{r}_0 \ldots, \underset{\sim}{r}_{n-1} \rangle$;

(c) $\underset{\sim}{\sigma}_n$ is the following P_n-name: the first $i \in N$ such that $\bar{q}^{\gamma(n)}_i \restriction n$ is in the generic set of P_n if there is such an i, and i^* otherwise. So by (a), $\langle \underset{\sim}{r}_0, \ldots, \underset{\sim}{r}_{n-1} \rangle \Vdash^{P_n}$ "$\underset{\sim}{\sigma}_n < i^*$";

(d) $\langle \underset{\sim}{r}_0, \ldots, \underset{\sim}{r}_{n-1} \rangle \Vdash^{P_n}$ "$\bar{q}^{\gamma(n)}_{\sigma_n} \leq r^*_n$".

It is easy to see that if we succeed, $\bar{r} = \langle \underset{\sim}{r}_0, \underset{\sim}{r}_1, \ldots \rangle \in P$, and for each $\alpha \in N$, for some $l < \omega$, $\alpha = \gamma(l)$, and so $\bar{r} \Vdash^P$ "$\sigma_l \in N$ and $\bar{q}^{\gamma(l)}_{\sigma_l}$ will be in the generic set".

So we have to do the induction step. For $n = 0$ there is no problem. So suppose r^{n*}, $\underset{\sim}{r}_0, \ldots, \underset{\sim}{r}_{n-1}$, $\underset{\sim}{\sigma}_0, \ldots, \underset{\sim}{\sigma}_{n-1}$ are defined, as we shall define $\underset{\sim}{r}_n$, $\underset{\sim}{\sigma}_n$.

We define $\underset{\sim}{\sigma}_n$, r^*_n by (b)-(d). Now let G_n be a generic set for P_n to which $\langle \underset{\sim}{r}_0, \ldots, \underset{\sim}{r}_{n-1} \rangle$ belongs, and we work for a while in $V[G_n]$. So now $\sigma_0, \ldots, \sigma_n$ are ordinals $< i^*$, in fact $\in N$. So $Q_n \in N[G]$ and $I^n_\alpha = \{\bar{q}^\alpha_i(n) : \bar{q}^\alpha_i \restriction n \in G_n\} \in N[G]$ as well as $\langle I^n_\alpha : \alpha < \alpha_0 \rangle$, and I^n_α is predense in Q_n.

Now in $M_0[G_n]$ there is a model M^1 with universe i^* (and countable language) with Skolem functions w.l.o.g. such that if $N^1 \prec M^1$, $\|N^1\| = \aleph_0$, $|N^1| \cap \omega_1$ is an ordinal, then for some $r \in Q_n$, $\{\bar{q}^\alpha_i(n) \in I^n_\alpha : i \in N^1\}$ is predense over r for every $\alpha \in N^1$.

Let $F = \langle F_m : m < \omega \rangle$ be the functions of N^1, so $\mathbf{M}^1$, $\mathbf{F} \in N$ (more exactly such names are in N), so for every $\bar{a} \in i^* \cap N$, $n < \omega$, $\{p \in P_n : p \Vdash \mathbf{F}_n(\bar{a}) = \beta$ for some $\beta\}$ is predense in P_n, belongs to N. Hence its intersection with N is predense above $\langle \underset{\sim}{r}_0, \ldots, \underset{\sim}{r}_{n-1} \rangle$; hence $\langle \underset{\sim}{r}_0, \ldots, \underset{\sim}{r}_{n-1} \rangle \Vdash$ "$\mathbf{F}_n(\bar{a}) \in N$". So in $V[G_n]$, $|N| \cap i^*$ is the universe of an elementary submodel of M^1. As $\langle \underset{\sim}{r}_0, \ldots, \underset{\sim}{r}_{n-1} \rangle \in G_n$, also $r^*_n(n) \in Q_n$, and we can relativize the above discussion to $\{r \in Q_n : r \geq r^*_n(n)\}$.

So there is $r_n \in Q_n$, $r^*_n(n) \leq r_n$, above which $\{\bar{q}^\alpha_i(n) \in I^n_\alpha : i \in N\}$ is predense for $\alpha \in N$. Now $\underset{\sim}{r}_n$ (its name) is as required.

LEMMA 3.5. *Suppose P_0 is trivial, $P_{i+1} = P_i * Q'_i$ $(i < \alpha)$ for limit $\delta \leq \alpha$, P_δ is the inverse limit if* cf $\delta = \aleph_0$, *and the direct limit otherwise.*

If $\phi \Vdash^{P_i}$ "Q_i is proper" then P_α is proper.

PROOF. Like 3.4 (if $N \prec M_0$ in $N \cap \alpha$, there is a last element, trivial; otherwise there are $\gamma_n < \gamma_{n+1} \in N \cap \alpha$, unbounded in $N \cap \alpha$, and repeat the previous argument (proving simultaneously, by induction on α, that if p_{n+1} is a P_{γ_n}-name of a member of $P_{\gamma_{n+1}}$, $p_{n+1} \upharpoonright \gamma_n = \phi$ then there is q in P_δ above all p_n, i.e. $q \upharpoonright \gamma_n$ force $p_{n+1} \leq q$).

Conclusion 3.6. Assume $2^{\aleph_1} = \aleph_2$, $2^{\aleph_0} = \aleph_1$, then there is a forcing notion P, $|P| = \aleph_2$, satisfying the $\aleph_2$-chain condition, such that in V^P, for any proper Q, $|Q| = \aleph_1$, and dense sets $\mathscr{D}_i \subseteq Q$ $(i < \omega_1)$ there is $G \subseteq Q$ generic for the $\mathscr{D}_i$'s.

PROOF. We iterate with countable support ω_2 times forcings with universe ω_1. Then by 3.5, $\underline{P}$ is proper, so $\aleph_1$ is not collapsed. As to the $\aleph_2$-chain condition note that even if $|\underline{P}| = \aleph_1$, $V^{\underline{P}} \models |Q| = \aleph_1$, $P * Q$ is not of power $\aleph_1$ (as we have many names) however by the properness, above any condition there is a condition with countable history, and then we can apply the Δ-system argument. The rest is trivial.

REMARK. (1) For a full development of proper forcing see the forthcoming survey of E. Wimmers [W].

(2) If one is left unconvinced by the proof of the $\aleph_2$-chain condition, iterate κ times with countable support, κ the first strongly inaccessible.

§4. Now we combine the methods of §1 (S-chain condition) and §3 (properness) to prove Theorem 2.1.

DEFINITION 4.1. A forcing notion P is S-proper, for an $\aleph_1$-oracle S (see Definition 1.6) if for every $P' \subseteq P$, $|P'| \leq \aleph_1$, $p^* \in P$, there are P'', $P' \subseteq P''$, $|P''| = \aleph_1$, and a one-to-one function from P'' into ω_1, and $A \subseteq \omega_1$ such that

(*) if $A \cap \delta \in \mathscr{S}_\delta$ then there is $p \in P$, $p \geq p^*$, such that: if $S \in \mathscr{S}_\delta$, $S \subseteq \delta$, $f^{-1}(S)$ is predense in $f^{-1}(\delta)$, *then* $f^{-1}(S)$ is predense above p.

Claim 4.2. If P is S-proper then

(1) in Definition 4.1, we can replace f by any other one-to-one function f' from P'' into ω_1, and then we need to change A only;

(2) if S is a strong $\aleph_1$-oracle, then P is proper, provided that $|P| = \aleph_1$.

LEMMA 4.3. *Suppose* S *is a proper extension of* S′, *both* $\aleph_1$-*oracles and* P *are* S-*proper. Then in* V^P, $({}^{\omega>}2) \cap V$ *is still of the second category* (*and similarly for any* W *as in* 1.5).

PROOF. Like 1.10.

LEMMA 4.4. *Suppose* P *is* S-*proper*, $|P| \leq \aleph_1$. *Then in* V^P *there is an* $\aleph_1$-*oracle* S′ (*defined uniformly so we denote it by* S^P *or* $S[G]$ *if* $G \subseteq P$ *is generic*) *such that for any* S′-*proper* $Q \in V^P$, $P * Q$ *is* S-*proper. It is done in such a way that* $S^{P*Q} = (S^P)^Q$.

PROOF. Like 1.12. We assume the set of elements of P is ω_1. By the S-properness of P, and few manipulations for some $A \subseteq \omega_1$, $A \cap \delta \in \mathscr{S}_\delta$ implies that for every $p < \delta$, there is $p' > p$, such that: $S \in \mathscr{S}_\delta$, $S \subseteq \delta$, S predense in $P \cap \delta$ above p implies S is predense in P above p'. Let $G \subseteq P$ be generic, then clearly

$$U = \{\delta < \omega_1 \colon \delta \text{ limit and for some } p_\delta \in G, \textit{ if } S \in \mathscr{S}_\delta,\ S \subseteq \delta \text{ is predense in } P \cap \delta \text{ then it is predense in } P \text{ above } p_\delta \text{ (hence } S \cap G \neq 0)\}$$

is nonempty. Moreover if $B \subseteq \omega_1$, $B \in V$, $B \neq \emptyset$ mod $\mathscr{D}_S$, then $B \cap U \neq \emptyset$. We define $\mathscr{S}'_\delta = \mathscr{S}_\delta[G \cap \delta]$ for $\delta \in U$, and arbitrarily otherwise. Now $S' = \langle \mathscr{S}'_\delta \colon \delta < \omega_1 \rangle$ shows our conclusion. Note that $G \cap \delta$ is generic for $P \upharpoonright \delta$ in $\mathscr{S}_\delta$, for every $\delta \in U$, so $\mathscr{S}'_\delta$ is of the right form.

Also S has the "diamond property". For if $B \subseteq \omega_1$ in $V[G]$, $\underset{\sim}{B} \in V$ is a name. Let $I(\underset{\sim}{B}, \alpha)$ be $\{p: p \Vdash^P \alpha \in \underset{\sim}{B}$ or $p \Vdash^P \alpha \notin \underset{\sim}{B}\}$. Clearly $I(\underset{\sim}{B}, \alpha)$ is predense, and (in V)

$$C = \{\delta: \text{for every } \alpha < \delta, I(\underset{\sim}{B}, \alpha) \cap \delta \text{ is predense in } P|\delta\}$$

is closed unbounded. So $\delta \in C \cap U$ implies $A \cap \delta \in \mathscr{S}_\delta[G \cap \delta] = \mathscr{S}'_\delta$.

We leave the rest to the reader.

LEMMA 4.5. *Suppose $P_{n+1} = P_n * Q_n$, P_0 trivial, Q_n is S^{P_n}-proper for $n < \omega$. Then the inverse limit P of the P_n is* S*-proper.*

PROOF. Like 3, 4, with a little more checking.

LEMMA 4.6. *Suppose P_0 is trivial, $P_{i+1} = P_i * Q_i$ $(i < \delta)$ for limit $\delta \leq \alpha$, $\operatorname{cf} \delta = \omega$, P_δ is the inverse limit of P_i $(i < \delta)$, and for limit $\delta \leq \alpha$, $\operatorname{cf} \delta > \omega$, P_δ is the direct limit of P_i.*

If Q_i is S^{P_i}-proper $|Q_i| = \aleph_1$, then P_α is S*-proper.*

PROOF. Like 3.5.

Now we return to Theorem 2.1.

PROOF OF THEOREM 2.1. Clearly it suffices to prove 2.4 and assume $V = L$. It suffices to prove:

(*) For an oracle S, and $A, B \subseteq {}^\omega\omega_1$ as in 2.4(3) there is an S-proper forcing Q, $|Q| = \aleph_1$,

$$\Vdash^Q (A \cup {}^{\omega>}\omega_1, \prec) \cong (B \cup {}^{\omega>}\omega_1, \prec).$$

For then we iterate the forcing Q_i, P_i $(i < \omega_2)$, as in 4.6. $|P_i| = \aleph_1$ each time dealing with one isomorphism, and so $\bigcup_{i<\omega_2} P_i$ (as in the proof of 3.6 it satisfies the $\aleph_2$-chain condition) is as required in 2.4. Hence prove 2.1 (by 2.4).

PROOF OF (*). We let Q^* be the set of partial isomorphisms from $(A \cup {}^{\omega>}\omega_1, \prec_1^{\dagger})$ onto $(B \cup {}^{\omega>}\omega_1, \prec_1^{\dagger})$, $\operatorname{Dom} f = X \cap (A \cup {}^{\omega>}\omega_1)$, such that for some limit or zero $\alpha < \omega_1$, and finite $A \subseteq {}^{\omega>}\omega_1$, $f \upharpoonright (\operatorname{Dom} f - A)$ is an isomorphism from ${}^{\omega>}\alpha \cup ({}^\omega\alpha \cap A)$ onto ${}^{\omega>}\alpha \cup ({}^\omega\alpha \cap B)$. The α is unique and we denote it by $\alpha(f)$.

Trivially for $\eta \in A \cup {}^{\omega>}\omega_1$, $\{f \in Q^*: \eta \in \operatorname{Dom} f\}$ is dense in Q^*, and for $\eta \in B \cup {}^{\omega>}\omega_1$, $\{f \in Q^*: \eta \in \operatorname{Range} f\}$ is dense in Q^*. So we have to prove only that Q^* is S-proper. For this we do a preliminary S-proper forcing (iterating ω_1 times) ensuring that

(*) Let $\delta < \omega_1$ be limit, $P^0 \subseteq Q^*$ be countable, and suppose:

(a) for every $f \in P^0$ for some $\alpha < \delta$, $(\operatorname{Dom} f) \cup (\operatorname{Range} f) \subseteq {}^{\omega\geq}\alpha$;

(b) every finite $f \in Q^*$, $(\operatorname{Dom} f) \cup (\operatorname{Range} f) \subseteq {}^{\omega>}\delta$ is in P^0;

(c) if $f, g \in P^0$, $f \cup g \in Q^*$ then $f \cup g \in P^0$;

(d) if $f \in P^0$, $\alpha < \delta$, then for some $g, f \subseteq g \in P^0$, $\alpha(g) \geq \alpha$.

Then there is $f \in Q^*$, $\alpha(f) = \delta$, such that for every $\alpha < \delta, f \upharpoonright ({}^{\omega\geq}\alpha) \in P^0$.

To ensure one instance of (*) we use 1.5 (we have to replace ${}^\omega\omega$ by ${}^\omega\delta$, but this is trivial). So after ω_1 times "we catch our tail", i.e. consider all possible P^0's (possible by the S-properness, and we can iterate by the previous lemmas, or even §1) as the S-chain condition implies S-properness).

Now (*) trivially ensures that Q^* is S-proper for every oracle S, so we are finished.

THEOREM 4.7. *It is consistent with ZFC* + $2^{\aleph_0} = \aleph_2$ (*if ZFC is consistent*) *that there is a universal* (*linear*) *order of power* $\aleph_1$.

REMARKS. The situation for the existence of saturated model of a first-order theory T of power λ, is known, see [Sh 1, VIII 4.7]. As for universal models, is was known that $\lambda = {}^{\lambda >} 2 > |T|$ is sufficient, and it was natural to assume that for unstable theories this condition is also necessary. Note that if we add $\aleph_2$ Cohen reals there is no universal order of power $\aleph_1$.

Problem. (1) For which T can we prove the analog of 4.7?

(2) Is there a condition ensuring T has a universal model of power λ iff $\lambda = {}^{\lambda >}2$?

(3) Generalize 4.7 to larger cardinals, even $> 2^{\aleph_0}$.

DEFINITION 4.8. Let K be the class of orders M of power $\aleph_1$ satisfying:

(1) M has cofinality $\aleph_1$, moreover for every a $\in M$, $\{b : b < a\}$ has cofinality $\aleph_1$.

(2) $M = \bigcup_{i<\omega_1} M_i$, M_i increasing, continuous, each M_i has the order type of the rationals, and for every $a \in M - M_i$, $\{b \in M$: $(\forall x \in M_i)$ $(a < x \equiv b < x)\}$ has confinality $\aleph_1$ and the set of Dedekind cuts of M_i realized by members of $M - M_i$, is of the second category.

(1)*, (2)* The same as (1), and (2) for the inverse order.

PROOF OF 4.7. Clearly every order of power $\aleph_1$, can be embedded into a member of K, provided that there is a set of reals of the second category of power $\aleph_1$. So we imitate the proof of 2.4 (in §4), making any two members of K isomorphic. Given M, $N \in K$, let $M = \bigcup_{i<\omega_1} M_i$, $N = \bigcup_{i<\omega_1} N_i$ so that $\langle M_i : i < \omega_1 \rangle$, $\langle N_i : i < \omega_1 \rangle$ are as required in (2), and (2)* simultaneously. We want to use $Q^* = \{f : f$ as an isomophism from M_i onto N_i, for some $i < \omega_1$, such that: for every $x \in M - M_i$ there is $y \in N - N_i$ and for every $y \in N - N_i$ there is $x \in M - M_i$ such that $(\forall a \in M_i)(a < x \equiv f(a) < y)\}$.

Again we have to do preliminary forcing to make Q^* proper.

REFERENCES

[B] J. BAUMGARTNER, *All $\aleph_1$-dense sets of reals can be isomorphic*, ***Fundamenta mathematicae***, vol. 79 (1973), pp. 101–106.

[Sh 1] S. SHELAH, ***Classification theory and the number of non-isomorphic models***, North-Holland, Amsterdam, 1978.

[Sh 2] ———, *Models with second-order properties*. III, *Omitting types in λ^+ for $L(Q)$*, ***Proceedings of the Berlin Workshop, July 1977, Archiv für Mathematische Logik*** (to appear).

[Sh 3] ———, *It is consistent that $\mathscr{P}(\omega)$ / finite has no non-trivial automorphisms* (preprint).

[Sh 4] ———, *Whitehead problem, independence of categoricity simple theories and Boolean algebras*, ***Notices of the American Mathematical Society***, vol. 25 (1978), A-441.

[Sh 5] ———, *Iterated forcing and independence results*, ***Notices of the American Mathematical Society***, vol. 25 (1978), A-497.

[ST] R.M. SOLOVAY and S. TENENBAUM, *Iterated Cohen extensions and Souslin's problem*, ***Annals of Mathematics***, vol. 94 (1971), pp. 201–245.

[W] E. WIMMERS, *The Shelah P-point independence theorem*, ***Israel Journal of Mathematics*** (to appear).

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