4. Classical forcing notions.

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We introduce basic examples of forcings, which were studied together with the extensions they determine, as one of the first. Namely Cohen, Random (Solovay), Hechler, Mathias, Laver, Miller and Sacks forcing. We show some phenomena that could arise in particular generic extensions, we focus on those phenomena that can be expressed only using semisets, i.e. we work only in $(\mathcal{P}^{V[G]}(V)) \subset V[G]$. We suppose throughout this chapter that in groundmodel Axiom of choice holds true.

4.1 Reading of names. Given a forcing notion $(P, \leq)$ we have worked with different relations $r \subseteq P \times V$ and when we had a generic filter G on P over V we considered semisets of the form $r[G]$. It is natural to look at the relation r as an approximation of this semiset. This approximation is a set from the groundmodel, in our case V. The generic filter then 'interprets' or 'specifies' the exact meaning of r. In view of this, we consider the relations to be 'names' for semiset. We shall adopt the convention to 'dot' the relations, i.e. write $\dot r$ instead of r, when we want to emphasize that we are thinking of them as names. We shall also write $\dot r/_G$ instead of $r[G]$.

Given a groundmodel set $x \in V$, we shall sometimes want to have a name for x, customarily denoted by $\check x$. We could either choose $\check x$ to be some canonical relation $\subseteq P \times x$ or, as we shall do here, we identify $\check x$ with x and define $\check x/_G$ to be x.

4.2 Boolean names. We have already mentioned, that the difference between an ordering P and the complete Boolean algebra $RO(P)$ is irrelevant from the point of view of forcing. However, working with a Boolean algebra can sometimes be more convenient, since we can use the Boolean operations. We shall now describe how to pass from P-names, i.e. relations $r \subseteq P \times V$, to appropriate B-names (see also 5.4).

Starting with a name $r \subseteq P \times V$ we can consider the following modifications

$$\tilde r = \{\langle q, x\rangle : (\exists p \geq q)(\langle p, x\rangle \in r)\}$$

and

$$\mathring r = \{\langle p, x\rangle : x \in \mathrm{rng}(r) \ \&\ \{q \in P : \langle q, x\rangle \in \tilde r\} \text{ is dense below } p\}.$$

We immediately see that $\mathrm{dom}(\tilde r) = \bigcup\{(\leftarrow, p] : p \in \mathrm{dom}(r)\}$ so $\mathrm{dom}(\tilde r)$ is downwards closed and $\tilde r[\{q\}] = \bigcup\{r[\{p\}] : q \leq p\}$. Also, recalling that a set $E \subseteq P$ is predense below $p \in P$ if there is no $q \leq p$ which would be disjoint from E, i.e. $q \in E^{\perp}$, we see that $\langle p, x\rangle \in \mathring r$ iff $r^{-1}[\{x\}]$ is predense below p.

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4.3 FACT. *It is clear that $r \subseteq \tilde{r} \subseteq \mathring{r}$ and, moreover,*

(i) $\mathring{r}^{-1}[\{x\}]$ *is a regular subset of* P *for all* x *(see 2.46).*

(ii) For any generic filter G *on* P*, the interpretations are the same, i.e.* $r[G] = \tilde{r}[G] = \mathring{r}[G]$.

The proof is straightforward. Note that in (ii) genericity of the filter G is only needed for the last equality, the first holds for any filter.

Using (i) and the fact that regular subsets of P form a complete Boolean algebra denoted $RO(P)$ (see 2.43) a relation $r \subseteq P \times V$ naturally determines a function $f : rng(r) \to RO(P)^+$, namely:

$$f(x) = \mathring{r}^{-1}[\{x\}].$$

For this function we have that for any generic filter G on P

$$r[G] = \{x : f(x) \in \bar{G}\},$$

where $\bar{G} = \{X \in RO(P)^+ : X \cap G \neq \emptyset\}$.

4.4 RELATIONSHIP BETWEEN NAMES AND CONDITIONS. We shall now introduce a piece of standard notation (see 5.13 for more details): Given a name $\dot{r} \subseteq P \times V$, $x \in V$ and a condition $p \in P$ we shall write

$$p \Vdash x \in \dot{r} \text{ or, more precisely } p \Vdash \check{x} \in \dot{r}$$

iff $\langle p, x\rangle \in \mathring{r}$. It is said that 'the condition p knows that x is an element of $\dot{r}$' or 'p forces $\check{x} \in \dot{r}$.'.

What about p forcing that x is *not* in $\dot{r}$. We might be tempted to that $p \Vdash \check{x} \notin \dot{r}$ means $\langle p, x\rangle \notin \mathring{r}$. It turns out, however, that this would not work in general because $\langle p, x\rangle \notin \mathring{r}$ might mean that p just doesn't know whether $\check{x} \in \dot{r}$. The correct definition will be

$$p \Vdash \check{x} \notin \dot{r} \text{ iff } p \text{ is disjoint from all elements of } r^{-1}[\{x\}],$$

i.e. $p \in \{q \in P : q \Vdash \check{x} \in \dot{r}\}^{\perp}$.

So in general, given a condition $p \in P$, a name $\dot{r}$ and an element $x \in V$ there are three possibilities:

(i) $p \Vdash \check{x} \in \dot{r}$

(ii) $p \Vdash \check{x} \notin \dot{r}$

(iii) p does not know, i.e. there are (disjoint) elements $q_1, q_2 \leq p$ such that $q_1 \Vdash \check{x} \in \dot{r}$ and $q_2 \Vdash \check{x} \notin \dot{r}$.

The generic filter G allows us to evade three valued or even intuitionistic logic since whenever $p \in P \cap G$ such that p does not decide $\check{x} \in \dot{r}$ the set

$$H = \{q \leq p : q \Vdash \check{x} \in \dot{r} \ \vee \ q \Vdash \check{x} \notin \dot{r}\}$$

is open dense below p so there is $q \in G \cap H$ below p which decides $\check{x} \in \dot{r}$.

4.5 THE ROLE OF DENSE SUBSETS. Consider a forcing notion P with a dense subset $H \subseteq P$. In the previous chapter we have said that it does not matter whether we work with P or H. However, given a P-name $\dot{r}$ it is not immediately clear that this is an H-name. In fact, it need not be, since, e.g. $dom(r) \cap H$ can be empty. To overcome this, we pass to $\tilde{r}$ for which we necessarily have $dom(\tilde{r}) \cap H \neq \emptyset$ (provided $dom(r)$ was nonempty). Moreover it turns out that for each $h \in H$, $x \in V$ we have

$$h \Vdash \check{x} \in \dot{r} \text{ iff } h \Vdash \check{x} \in \tilde{r}$$

An easy density argument now shows that $\dot{r}$ and $\dot{s} = \tilde{r} \cap H \times dom(r)$ are names for the same semiset, i.e. $\dot{r}/_G = \dot{s}/_G$ for each generic filter G.

4.6 EQUALITY OF NAMES. When we will be trying to prove the consistency of the failure of CH, we will do this by adding semisets which are subsets of ω. For this to work we have to add a lot of different semisets, however we only have names for the semisets and we know, that different names can be actually names for the same semiset. So it will be important to know, when two names are really names for different semisets. We first start with the opposite question: When are two names $\dot{r}$ and $\dot{s}$ names for the same semiset? The answer is that a condition $p \in P$ 'knows' that the two P-names $\dot{r}, \dot{s}$ are actually equal, i.e. $\dot{r}/_G = \dot{s}/_G$ for any generic G containing p, iff for any $x \in \mathrm{rng}(r) \cup \mathrm{rng}(s)$ one of the following equivalent conditions is met:

(i) $(\forall q \leq p)(r^{-1}[\{x\}]$ is predense below $q \leftrightarrow s^{-1}[\{x\}]$ is predense below $q)$,

(ii) $(\forall q \leq p)(q \perp r^{-1}[\{x\}] \leftrightarrow q \perp s^{-1}[\{x\}])$.

If one of these conditions is met, we write $p \Vdash \dot{r} = \dot{s}$ and we write $P \Vdash \dot{r} = \dot{s}$ if all conditions force $\dot{r}$ and $\dot{s}$ to be equal, i.e. iff $(\forall x \in \mathrm{dom}(r) \cup \mathrm{dom}(s))((r^{-1}[\{x\}])^{\perp} = (s^{-1}[\{x\}])^{\perp})$. This implies that, necessarily, $\mathrm{rng}(r) = \mathrm{rng}(s)$.

4.7 EXAMPLES. **(i)** We say that P forces $\check{x} \in \dot{r}$ if each $p \in P$ forces $\check{x} \in \dot{r}$. Clearly $P \Vdash \check{x} \in \dot{r}$ iff $r^{-1}[\{x\}]$ is predense in P and $P \Vdash \check{x} \notin \dot{r}$ iff $r^{-1}[\{x\}] = \emptyset$.

(ii) Given two names $\dot{r}, \dot{s}$ then

(a) $p \Vdash \dot{r} = \dot{s}$ iff $(\forall x \in \mathrm{rng}(r) \cup \mathrm{rng}(s))(\forall q \leq p)(q \Vdash \check{x} \in \dot{r} \leftrightarrow q \Vdash \check{x} \in \dot{s})$,

(b) $p \not\Vdash \dot{r} = \dot{s}$ iff $(\exists q \leq p)(\exists x)(q \Vdash \check{x} \in \dot{r}\ \&\ q \Vdash \check{x} \notin \dot{s} \ \vee\ q \Vdash \check{x} \in \dot{s}\ \&\ q \Vdash \check{x} \notin \dot{r})$,

(c) $p \Vdash \dot{r} \neq \dot{s}$ iff $(\forall p_1 \leq p)(\exists q \leq p_1)(q \not\Vdash \dot{r} = \dot{s})$.

(iii)

(a) $P \Vdash \dot{r} \subseteq \check{\omega}$ iff $\mathrm{rng}(r) \subseteq \omega$

(b) $p \Vdash$ "$\dot{r} \subseteq \check{\omega}\ \&\ \dot{r}$ is infinite" iff $\tilde{r}[(\leftarrow, p)] \subseteq \omega$ for all $m < \omega$ the set $r^{-1}[(\omega \setminus m)]$ is predense below p iff for any generic filter G with $p \in G$ the semiset $\dot{r}/_G$ is an infinite part of ω.

(c) $p \Vdash$ "$\dot{r} \subseteq \check{\omega}\ \&\ \dot{r}$ is finite" iff $\tilde{r}[(\leftarrow, p)] \subseteq \omega$ and $(\forall p_1 \leq p)(\exists q \leq p_1)(\exists k < \omega)(q \in (r^{-1}[(\omega \setminus k)])^{\perp})$.

(iv) $P \Vdash \dot{r} : \check{\omega} \to \check{\omega}$, i.e. P forces that $\dot{r}$ is a mapping of ω into ω. When we define a mapping

$$r^{\heartsuit}(p) = \bigcup r[\,[p, \to)\,] \qquad \text{for each } p \in P,$$

then $r^{\heartsuit}$ maps P into partial functions from ω into ω and for any $n \in \omega$ the set

$$\{p \in P : n \in \mathrm{dom}(r^{\heartsuit}(p))\}$$

is open dense in P.

Now suppose that $B \approx RO(P)$ represents regular subsets of P 2.46. Then the name $\dot{r}$ determines an (ω, ω)-matrix, namely:

$$a(n, m) = \{p \in P : p \Vdash \langle n, m\rangle \in \dot{r}\} = \{p \in P : \langle n, m\rangle \in r^{\heartsuit}(p)\}.$$

Moreover, when $P \Vdash$ "$\dot{r} : \check{\omega} \to \check{\omega}$ is surjective mapping", then $\{p \in P : m \in \mathrm{rng}(r^{\heartsuit}(p))\}$ is dense for each $m \in \omega$ and it follows that the matrix $a(n, m)$ is surjective 2.58.

We bring out the standard definition of particular forcing notion and if possible we also give alternative description using subsets of the Cantor space $\mathcal{C}$ or Baire space $\mathcal{N}$.

First we start with notation and remind well known elementary facts concerning trees, σ-fields of Borel sets and sets with Baire property in Cantor $\mathcal{C}$ and Baire $\mathcal{N}$ spaces and their quotients and some basics of measure theory.

4.8 NOTATION. *Symbols $\exists^\infty$ and $\forall^\infty$ are abbreviations of 'there is infinitely many' and 'for all except finitely many' respectively.*

Let r be a binary relation, instead of $r[\{x\}]$ we use common abbreviation $r''\{x\} = \{y \in rng(r) : \langle x, y\rangle \in r\}$, Similarly we denote $r[G]$ as $r''G$.

Majority of forcing notions introduced in this chapter has a tree structure.

4.9 PROPERTIES OF TREES. We call a tree T a *binary tree* if it is a subtree of $(Seq(2), \subseteq)$, or *ω-ary* is it is a subtree of $(Seq, \subseteq)$, note that always $\emptyset \in T$. We are mainly interested in trees that have all branches infinite.

(i) $t \in T$ is splitting node if there exists $i \neq j$ such that $t^\frown i, t^\frown j \in T$; i.e. t has at least two immediate successors. For a splitting node we define splitting set $splt(t) = \{i \in \omega : t^\frown i \in T\}$.

(ii) Stem of a tree T is the splitting node of smallest rank.

(iii) If $s \in T$ then $T(s) = \{t \in T : t \subseteq s \text{ or } t \supseteq s\}$ is a subtree of T.

(iv) $[T]$ is the set of all infinite branches of T.

There is a close correspondence between trees and closed sets.

4.10 FACT. *(i) It T is a binary tree, then $[T]$ is closed subset of Cantor space $\mathcal{C}$.*

*(ii) If $\emptyset \neq U \subset \mathcal{C}$ is a closed set, then $T_U = \{f \restriction n : f \in U,\ n \in \omega\}$ is a tree and $[T_U] = U$.
The similar statement holds for Baire space $\mathcal{N}$.*

Similarly to closed subsets of Cantor (or Baire) space one can characterise close nowhere dense subsets by trees as follows

4.11 DEFINITION. Tree T is called *nowhere dense* if satisfies $(\forall f \in T)\ \exists g \in T,\ f \subseteq g$, such that for each $h \in T$, $g \subseteq h$, h is not a splitting node of T.

Nowhere dense trees correspond to nowhere dense subsets.

4.12 BASES OF THE SPACE $\mathcal{C}$. For $t \in Fn(\omega, \{0, 1\})$ denote $[t] = \{f \in \mathcal{C} : f \supset t\}$.

(a) Let
$$\mathcal{B}_1 = \{[t] : t \in Fn(\omega, \{0, 1\})\}.$$
$\mathcal{B}_1$ is a base for topology of $\mathcal{C}$ and mapping $t \mapsto [t]$ is an isomorphism of $(Fn(\omega, \{0, 1\}), \supseteq)$ onto $(\mathcal{B}_1, \subseteq)$.

(b) Let $\mathcal{B}_2 = \{[s] : s \in Seq(2)\}$. $\mathcal{B}_2$ is also a base for topology of $\mathcal{C}$. Then $\mathcal{B}_1 = \{[k \oplus s] : s \in Seq(2)\ \&\ k \in \omega\}$, where $k \oplus s \in Fn(\omega, \{0, 1\}; \omega)$, $dom(k \oplus s) = [k, k + |s|)$ and $(k \oplus s)(k + i) = s(i)$, for each $i < |s|$.; i.e. $(k \oplus s)$ is a *shift* of s.

4.13 REALS. Phrase 'Forcing notion *adds a new real*' means that for any generic filter G over V, the generic extension $V[G]$ contains a new subset $\sigma \subset \omega$ of natural numbers and hence $V[G]$ contains a function $\rho : \omega \to \omega$ which does not belong to groundmodel V. In our terminology these are proper semisets of V that are obtained as $r''G$ for some relation $r \in V$.

It is quite common in set theory that under the term 'real' we mean subset of ω. Hence elements of Cantor space $\mathcal{C} = {}^{\omega}\{0,1\}$ are reals as well as the function from ω to ω, i.e. elements from Baire space $\mathcal{N}$ are called reals. The following definition reveals an interplay between new reals and groundmodel reals.

4.14 DEFINITION. Let M denote an extension of V.

(i) $X \subseteq \omega$ in the extension is said to be an *independent* (or *splitting*) *real* over V if for all $Y \in [\omega]^{\omega} \cap V$ both $X \cap Y$ and $Y - X$ are infinite.

(ii) A function $f \in M$, $f \in \omega^{\omega}$, is a *dominating real* over V if for all $g \in \omega^{\omega} \cap V$ for all but finitely many $n \in \omega$, $g(n) \leq f(n)$.

(iii) A function $h \in \omega^{\omega}$ in the extension is said to be an *unbounded real* over V if for all $f \in \omega^{\omega} \cap V$ the set $\{n \in \omega : h(n) > f(n)\}$ is infinite.

(iv) A function $h \in \omega^{\omega}$ in the extension is said to be an *eventually different* real over V if for all $f \in \omega^{\omega} \cap V$ the set $\{n \in \omega : h(n) = f(n)\}$ is finite.

(v) M is an ω^{ω}*-bounding extension* of V if every $f \in M$, $f \in \omega^{\omega}$ is dominated by a $g \in \omega^{\omega} \cap V$, i.e. $f(n) \leq g(n)$ for any n, which is equivalent to $\forall f \in \omega^{\omega}\ \exists g \in \omega^{\omega} \cap V$ such that $f \leq^* g$.

4.15 REMARK. Each dominating real is eventually different, hence if forcing adds a dominating real it automatically adds an eventually different real.

4.16 EXAMPLES OF FORCING NOTIONS. Each of the following forcing notions are atomless and are constructed so that the generic extension will contain a new real. Note that the conditions of each example are made so it is a finite approximation of desired new subset of reals and generic filter picks up some of them and assembles a new real.

All presented examples preserves cardinal number ω_1 in V, i.e. in no generic extension there is a mapping of ω onto ω_1. These examples are moreover 'proper forcing' - notion that will be mentioned later, or see [Jec02] or [She98].

On a classical examples of forcings we illustrate

(a) some phenomena that should arise in generic extension,

(b) a profitability of using equivalent descriptions of a particular forcing notion,

(c) some elementary techniques and ideas that should make the general statements easier to understand; the use of Boolean matrices; absoluteness of cardinal numbers in generic extensions; some forms of distributivity; Laver and Sacks property and some others

In the Definition 3.3 there were introduced a forcing equivalence. What does this equivalence means to generic extension? If orderings P, Q are equivalent $P \sim Q$, then for each generic filter G on P over V, there is generic filter G' on Q over V such that G and G' are similar and so $V[G] = V[G']$: Since G and G' are similar there are relations r_1 and r_2 in V such that $r_1''G = G'$ and $r_2''G' = G$, so for each relation $s \in V$ we get $s''G = (r_2 \circ s)''G'$, where $r_2 \circ s = \{\langle x, y\rangle : (\exists z)\langle x, z\rangle \in r_2\ \&\ \langle z, y\rangle \in s\}$.

CONCLUSION. When investigating generic extension one can use any equivalent of forcing notion and can skip from one equivalent to another according the demand.

4.17 DEFINITION. Cohen forcing. *Cohen forcing* is countable atomless ordering and is equivalent to any of the following set

(a) $Seq = \bigcup\{^n\omega : n < \omega\}$,

(b) $Seq_2 = \bigcup\{^n 2 : n < \omega\}$,

(c) $Fn(\omega, 2) = \{f; f : D \to \{0,1\}, D \in [\omega]^{<\omega}\}$,

ordered by inverse inclusion $\supseteq$.

Each of this ordered sets is separative and homogeneous partial order. Since $(Fn(\omega, 2), \supseteq)$ is isomorphic to base $\mathcal{B}_1$ (4.12) of the space $\mathcal{C}$, complete Boolean algebras of orderings *(a) - (c)* are isomorphic to $RO(\mathcal{C}) \approx$ BOREL$(\mathcal{C})/$*Meagre*$(\mathcal{C})$. This algebra have simple description.

4.18 DEFINITION. Cohen algebra. A complete, atomless ccc Boolean algebra with countable dense set is called *Cohen algebra* and is usually denoted by **C**, or $\mathbf{C}(\omega)$.

We already know, that any two Cohen algebras are isomorphic (they have isomorphic dense subset) and they are homogeneous, see 2.72. Cohen algebra is ccc since it contains a countable dense subset, i.e. in fact it is σ-centred.

4.19 EXAMPLE. For any perfect Polish space S algebras $RO(S) \approx$ BOREL$(S)/\mathcal{M}$ (2.53) are Cohen algebras since they have a countable base.

Subfamily of Borel sets of a perfect Polish space S, BOREL$(S)-$*Meagre*(S) ordered by inclusion is also useful representation of Cohen forcing. This ordering is not separative, but its separative quotient is exactly BOREL$(S)/$*Meagre*(S).

4.20 In the following part we show that Cohen forcing

(a) adds a new real,

(b) adds a splitting set,

(c) adds unbounded real,

(d) does not add an eventually different real, hence cannot add dominating reals.

4.21 GENERIC FILTER AND COHEN REAL. Fix a generic filter G on an ordering $(Seq(2), \supseteq)$, then for $s, t \in G$ is either $s \subseteq t$ or $s \supseteq t$. Put

$$\sigma_c = \bigcup\{s \in Seq(2) : s \in G\},$$
$$\rho_c = \{i \in \omega : (\exists s \in Seq(2)) \text{ such that } s(i) = 1 \;\&\; s \in G\}.$$

Both σ_c and ρ_c are called *Cohen real* over V.

4.22 LEMMA. *(i)* $\sigma_c, \rho_c \in V[G]$,

(ii) $\sigma_c : \omega \to \{0,1\}$,

(iii) $\rho_c \subset \omega$ *is infinite and splitting set.*

Proof. *(i)* Define relations $r_1 \subseteq Seq(2) \times (\omega \times \{0,1\})$ and $r_2 \subseteq Seq(2) \times \omega$.

$$\begin{aligned} r_1 &= \{\langle s, \langle i, s(i)\rangle\rangle \;:\; s \in Seq(2) \;\&\; i < dom(s)\;\}, \\ r_2 &= \{\langle s, i\rangle \;:\; s \in Seq(2) \;\&\; i < dom(s) \;\&\; s(i) = 1\;\} \end{aligned}$$

Then $\sigma_c = r_1'' G$ and $\rho_c = r_2'' G$. Moreover σ_c is a standard characteristic function of ρ_c.
(ii) Let $k \in \omega$ and put $H_k = \{s \in Seq(2) : k < |s|\}$. The set H_k is dense and so $\sigma_c : \omega \to \{0,1\}$, i.e. $dom(\sigma_c) = \omega$.
(iii) Let $Y \in [\omega]^\omega$. Since Y is infinite and $Y \in V$, following sets are dense for each $k \in \omega$ and $j \in \{0,1\}$

$$H^k_j = \{\; s \in Seq(2) \;:\; (\exists i \in Y - k)\;(i < dom(s)) \;\&\; s(i) = j\;\}.$$

It follows that both $Y \cap \rho_c$ and $Y - \rho_c$ are both infinite in $V[G]$. □

4.23 LEMMA. *Cohen forcing adds unbounded real.*

Proof. The lemma claims that in $V[G]$ there is $\tau : \omega \to \omega$ so that $(\exists^\infty i)\;\tau(i) > f(i)$, for each function $f : \omega \to \omega$ from the groundmodel.

Here we profit from the equivalence of orderings $(Seq, \supseteq)$ and $(Seq(2), \supseteq)$ from 4.17. Let $\bar{G}$ be a generic filter on $(Seq, \supseteq)$ similar to G and let $\tau = \bigcup\{t \in Seq : t \in \bar{G}\}$. Then $\tau \in V[\bar{G}] = V[G]$ and by similar arguments as in 4.22 it is a function from ω to ω.

For any $f \in {}^\omega\omega \cap V$ and $k \in \omega$ let

$$H^k_f = \{\; s \in Seq \;:\; (\exists i > k)\;(i < dom(s)) \;\&\; s(i) > f(i)\;\}.$$

Each set H^k_f is dense in Seq, which guarantees that τ is an unbounded function. □

4.24 LEMMA. *For any $\sigma \in V[G]$ $\sigma : \omega \to \omega$ there is $f \in {}^\omega\omega \cap V$ such that $\exists^\infty i \in \omega\; \sigma(i) = f(i)$; i.e. Cohen forcing does not add eventually different real.*

Proof. Take a Cohen algebra $\mathbf{C}$ and let G be a generic filter on $\mathbf{C}$ over V. Fix a countable dense subset $H \subset \mathbf{C}^+$ and fix an enumeration $H = \{h(n) : n \in \omega\}$. Let $\sigma \in V[G]$ $\sigma : \omega \to \omega$, then there is a matrix name for σ in V, i.e. a matrix $A = \langle a(m,n) \in \mathbf{C} : m, n \in \omega\rangle$. Define $f : \omega \to \omega$

$$f(m) = \min\{n \in \omega : a(m,n) \cap h(m) \neq \mathbf{0}\}.$$

Such mapping is well defined on ω since each $h(m) \neq \mathbf{0}$ and $\bigvee_n a(m,n) = \mathbf{1}$.
CLAIM. For any $k \in \omega$ $\bigvee_{m \geq k} a(m, f(m)) = \mathbf{1}$.

Suppose not, then $b = \mathbf{1} - \bigvee_{m \geq k} a(m, f(m)) \neq \mathbf{0}$ and since $\mathbf{C}$ is atomless, there is infinitely many $m \in \omega$ so that $h(m) \leq b$. But for such m we get $a(m, f(m))$ is compatible with b, which is in contradiction with the definition of f.

We get that for each $k \in \omega$ there is $m \geq k$ such that $a(m, f(m)) \in G$ and so $\tau(m) = f(m)$ for infinitely many m's in $V[G]$. □

4.25 DEFINITION. Cohen reals. Let M be an extension of V. We say that $\sigma \in M$, $\sigma \subseteq \omega$, or $\rho : \omega \to \{0,1\}$, or $\rho : \omega \to \omega$, $\rho \in M$ is a *Cohen real* over V if for characteristic function τ of σ the set $\{\tau \restriction n : n \in \omega\}$ is a generic filter on $(Seq(2), \supseteq)$ over V, or similarly $\{\rho \restriction n : n \in \omega\}$ is a generic filter on $(Seq(2), \supseteq)$, respectively on $(Seq, \supseteq)$ over V.

Now it is clear what we mean by the phrase "a forcing notion P adds a Cohen real"; i.e. any generic extension by P contains a Cohen real over groundmodel. The following lemma gives us a criterion for new real to be a Cohen.

4.26 LEMMA. *Let a mapping* $\rho : \omega \longrightarrow \{0, 1\}$ *be a semiset over* V*. Then* ρ *is a Cohen real if and only if for each nowhere dense binary tree* $T \in V$ *there is* $n \in \omega$ *such that* $\rho \restriction n \notin T$.

Proof. Simple density argument. □

4.27 EXAMPLE. Let $\rho : \omega \longrightarrow \{0, 1\}$ 4.21 be a Cohen real. Let $A \in [\omega]^\omega$, $A \in V$, be arbitrary and f any one-to-one mapping of ω onto A. Then $\tau : \omega \longrightarrow \{0, 1\}$ defined in V[G] by $\tau(n) = \rho(f(n))$ is a Cohen real.

Moreover if $\omega - A$ is infinite, then generic filter G_τ on $Seq(2)$ determined by τ is not similar to G therefore $V[G_\tau] \subsetneq V[G]$ and there are 2^ω-many such generic submodels of V[G].

4.28 ABSOLUTENESS OF CARDINALS. One of the first task concerning any forcing notion is whether it preserves cardinal numbers and cofinalities, i.e. whether cardinal numbers of the groundmodel are cardinal numbers in extension.

What does it mean to destroy cardinality of $\lambda_2 \in V$? It means that there is a cardinal $\lambda_1 \in V$ smaller then λ_2 and in the extension there is a mapping $\rho : \lambda_1 \longrightarrow \lambda_2$ that is onto.

4.29 THEOREM. *(i) Any* ccc *forcing preserves cardinals and cofinalities.*

(ii) Any $\kappa - cc$ *forcing preserves all cardinal numbers* λ, $\lambda \geq \kappa$ *and cofinalities* $\geq \kappa$.

Let us point out the core lemma

4.30 LEMMA. *Let* λ_1, λ_2 *be cardinals and* $\langle a(\alpha, \beta) \in B : \alpha < \lambda_1,\ \beta < \lambda_2 \rangle$ *be a matrix in a Boolean algebra* B*. If* B *is* κ-cc*, then the set* $S = \{\langle \alpha, \beta \rangle : a(\alpha, \beta) \neq \mathbf{0}_B\}$ *has cardinality at most* $|(\lambda_1 \times \kappa^-)|$*. Especially if* $\lambda_1 < \kappa$ *then* $|S| < \kappa$.

Proof. We can assume B is infinite and that κ is the least cardinal such that B is κ-cc. Such cardinal is uncountable and is a regular cardinal by result of Erdös and Tarski.

The set S is a union of λ_1-many rows each of size less then κ. If κ is not limit cardinal, there is its predecessor κ^- and $|S| \leq \max(\lambda_1, \kappa^-)$. Hence $|S| < \kappa$ provided that $\lambda_1 < \kappa$.

If κ is limit cardinal, then it is weakly inaccessible cardinal and $|S| < \kappa$ provided $\lambda_1 < \kappa$, $|S| \leq \lambda_1$ otherwise. □

Proof. PROOF OF THE THEOREM 4.29. Let P be κ-cc forcing and let $\lambda_1 < \lambda_2$ be cardinals such that $\kappa \leq \lambda_2$.

4.31 CLAIM. *If* $\rho : \lambda_1 \longrightarrow \lambda_2$ *is a mapping in extension* V[G]*, then there is a set* $A \in V$ *such that* $rng(\rho) \subseteq A$ *and either* $|A| < \kappa$ *if* $\lambda_1 < \kappa$ *or* $|A| = \lambda_1$ *otherwise. Therefore a mapping* ρ *cannot be a surjection.*

For ρ there is a matrix $\langle a(\alpha, \beta) : \alpha < \lambda_1,\ \beta < \lambda_2 \rangle$ in RO(P); a name for ρ, such that $\rho = \{\langle \alpha, \beta \rangle : a(\alpha, \beta) \in G\}$. It is sufficient to put $A = rng(S)$ for S from the preceding lemma. □

4.32 VIOLATING CONTINUUM HYPOTHESIS. When in groundmodel V the continuum hypothesis (CH) holds true, then it also holds true in the generic extension V[G] that is obtained by Cohen forcing.

The argument is easy, for each real in the extension we have a relation in the ground model and the number of possible relations is limited by the groundmodel continuum:

$$\mathcal{P}(\omega)^{V[G]} = \{r''G : r \subseteq Seq(2) \times \omega\}.$$

To violate CH, fix a cardinal number $\kappa > \omega_1$ e.g. ω_{13}.

4.33 DEFINITION. Forcing notions

(a) $(\mathrm{Fn}(\kappa \times \omega, \{0,1\}), \supseteq)$ and

(b) $(\mathrm{Fn}(\kappa, \{0,1\}), \supseteq)$

are isomorfic and called forcing notion for *adding* κ *many Cohen reals*.

These are homogeneous separative partial orderings satisfying ccc, see 2.21.

Any algebra isomorphic to $\mathrm{RO}(\mathrm{Fn}(\kappa, \{0,1\}), \supseteq)$ is called *Cohen algebra* and denoted by $\mathbf{C}(\kappa)$. Cohen algebra $\mathbf{C}(\kappa)$ is isomorphic to $\mathrm{RO}(2^\kappa) \approx \mathrm{BOREL}(2^\kappa)/\mathit{Meagre}$, where 2^κ is the generalized Cantor space of length κ.

Algebra $\mathbf{C}(\kappa)$ has dense set of size κ and $\mathfrak{g}_c(\mathbf{C}(\kappa)) = \kappa$. Characterisation of $\mathbf{C}(\kappa)$ is not so transparent as for $\mathbf{C} = \mathbf{C}(\omega)$, for more details see [BJZ97].

We will continue with ordering $(\mathrm{Fn}(\kappa \times \omega, \{0,1\}), \supseteq)$. Note that $(\mathrm{Fn}(\{\alpha\} \times \omega, \{0,1\}), \supseteq)$ is regular subordering of $(\mathrm{Fn}(\kappa \times \omega, \{0,1\}), \supseteq)$ for each $\alpha < \kappa$ and is isomorphic to Cohen forcing 4.17(c).

4.34 PROPOSITION. *Let* G *be a generic filter on* $(\mathrm{Fn}(\kappa \times \omega, \{0,1\}), \supseteq)$*, then*

(i) $\rho = \bigcup\{s \in \mathrm{Fn}(\kappa \times \omega, \{0,1\}) : s \in G\}$ *is a mapping* $\rho : \kappa \times \omega \longrightarrow \{0,1\}$*,*

(ii) $\rho_\alpha = \{\langle n, i\rangle : \rho(\langle \alpha, n\rangle) = i\}$ *is a Cohen real over* V*,*

(iii) if $\alpha \neq \beta$ *then* $\rho_\alpha \neq \rho_\beta$*.*

Proof. Density arguments work here. We show *(iii)*. Let $\alpha \neq \beta$. For each $k \in \omega$ the set

$$H_k = \{s \in \mathrm{Fn}(\kappa \times \omega, \{0,1\}) : (\exists i \in \omega)\ (i > k)\ \&\ \langle \alpha, i\rangle, \langle \beta, i\rangle \in \mathrm{dom}(s)\ \&\ s(\langle \alpha, i\rangle) \neq s(\langle \beta, i\rangle)\}$$

is dense. This implies that $\{i \in \omega : \rho_\alpha(i) \neq \rho_\beta(i)\}$ is infinite in $V[G]$. □

We already know that in $V[G]$ are cardinal numbers absolute and $V[G] \models 2^\omega \geq \kappa$ by 4.34(iii). If $\kappa^\omega = \kappa$ then $V[G] \models 2^\omega = \kappa$.

4.35 CONCLUSION. If ZFC is a consistent theory then CH is not provable within ZFC and the continuum can by arbitrary large.

4.36 RANDOM FORCING. One of the equivalents of Cohen forcing was $(\mathrm{BOREL}(2^\omega) - \mathit{Meagre}(2^\omega), \subseteq)$. Meagre sets are negligible sets from topological point of view. From measure theory point of view, negligible sets are those of measure zero. We consider standard probability σ-additive measure defined on a σ-field $\mathrm{BOREL}(\mathcal{C})$, which uniquely extends function $m : \mathcal{B}_1 \longrightarrow [0,1]$, $m([s]) = \frac{1}{2^{|s|}}$ and denote it by m.

4.37 FACT. *Let* $\emptyset \neq U \subset \mathcal{C}$ *is closed, then for a tree* $T(U) = \{f \restriction n : f \in U,\ n \in \omega\}$*,*

$$m(U) = \lim_{n \in \omega} \frac{1}{2^n} \cdot |T_n(U)|.$$

Therefore such measure is continuous (diffuse), see [Kec95], i.e. all singletons are of measure zero moreover m is normalised, $m = 1$. Let *Null* denote the σ-ideal of sets of measure zero. So $(\mathcal{C}, \mathrm{BOREL}(\mathcal{C}), m)$ is an example of probability space.

We will use the following classical results concerning measurable spaces where underlying set is a Polish space, for proofs see [Kec95].

4.38 THEOREM. *Let μ be a probability continuous Borel measure on a Polish space S. Then there is a Borel isomorphism $\varphi : \mathcal{C} \to S$, i.e. φ is one-to-one onto mapping that induces a measure preserving isomorphisms of the fields* $\mathrm{BOREL}(\mathcal{C})$ *and* $\mathrm{BOREL}(S)$.

4.39 EXAMPLE. Lebesgue measure. Our Borel measure m on the Cantor space corresponds to Lebesgue measure on the unit interval $[0,1]$ of real line. There is a continuous mapping

$$\begin{aligned} \varphi : \mathcal{C} &\longrightarrow [0,1] \\ f &\longmapsto \varphi(f) = \sum_{i=0}^{\infty} \frac{f(i)}{2^{i+1}}, \end{aligned}$$

which is onto and one-to-one with exception for dyadic rational numbers, where φ is 'two-to-one'. Nevertheless, for any $U \in \mathrm{BOREL}(\mathcal{C})$ is the set $\varphi[U]$ Borel in $[0,1]$ and $m(U)$ is equal to Lebesgue measure of $\varphi[U]$.

4.40 THEOREM. *Any continuous probability Borel measure μ on a Polish space S is regular (even Radon), i.e.*

$$\mu(A) = \inf\{\mu(U) : U \supseteq A,\ U \text{ open }\} \text{ and}$$

$$\mu(A) = \sup\{\mu(K) : K \subseteq A,\ K \text{ compact}\}.$$

4.41 DEFINITION. (i) $(\mathrm{BOREL}(\mathcal{C}) - \mathit{Null}, \subseteq)$ is *Random forcing*. The ordering is not separative, its separative quotient is

(ii) $(\mathrm{BOREL}(\mathcal{C})/\mathit{Null}, \subseteq)$. This is ccc complete atomless Boolean algebra that carries strictly positive σ-additive measure, $m[U] = m(U)$, for each $U \in \mathrm{BOREL}(\mathcal{C})$.

The following lemma is a corollary of Theorem 4.40.

4.42 LEMMA. *(i) The family of all closed sets in $\mathcal{C}$ of positive measure is a dense set in the Random forcing.*

(ii) Family of all G_δ sets in $\mathcal{C}$ of measure zero is a base of the ideal Null.

4.43 DEFINITION. A complete Boolean algebra B is called *measure algebra* if B carries a strictly positive probabilistic measure μ, i.e.

(i) $\mu : B \to [0,1]$,

(ii) $\mu(a) = 0$ iff $a = \mathbf{0}_B$ (μ is *strictly positive*),

(iii) for any disjoint family $\langle a_n : n \in \omega \rangle$

$$\mu(\bigvee\{a_n : n \in \omega\}) = \sum_{0}^{\infty} \mu(a_n),$$

i.e. μ is σ-additive,

(iv) $\mu(\mathbf{1}_B) = 1$, i.e. μ is *normalised*.

$(\mathrm{BOREL}(\mathcal{C})/\mathit{Null}, \subseteq)$ is particular but important example of so called *measure algebra* and is denoted $\mathbf{B}(\omega)$.

4.44 FACT. *Any measure algebra satisfies* ccc.

Proof. Let μ be a measure on B witnessing that B is a measure algebra. Let there be uncountably many disjoint elements in B. Then since μ is strictly positive, there are uncountably many elements among them with measure greater or equal to $\frac{1}{n_0}$, for some $n_0 \in \omega$. Using σ-additivity of measure we contradict the fact that μ is normalised. $\square$

4.45 EXAMPLE. $(\mathcal{P}(\omega), \subseteq)$ is a measure algebra. It is even complete field of sets - so it is a complete and atomic Boolean algebra. Let

$$\nu(A) = \sum\{\frac{1}{2^{n+1}} : n \in A\}, \text{ for each } A \subset \omega.$$

ν is probabilistic measure on $\mathcal{P}(\omega)$. This algebra is diametrically different from BOREL$(\mathcal{C})/$*Null* and is absolutely uninteresting from the forcing point of view.

4.46 In the following part we prove that Random forcing

(a) adds a new real,

(b) adds a splitting set,

(c) adds an eventually different real and

(d) it is a ω^ω-bounding forcing, hence cannot add unbounded reals.

Fix a generic filter G on BOREL$(\mathcal{C}) -$ *Null* over V.

4.47 GENERIC FILTER AND RANDOM REAL. Fix a generic filter G on a Random forcing. Put

$$\begin{aligned} \sigma_r &= \bigcup\{s \in Seq(2) : \langle s\rangle \in G\}, \\ \rho_r &= \{i \in \omega : (\exists s \in Seq(2)) \text{ such that } s(i) = 1 \ \& \ \langle s\rangle \in G\}. \end{aligned}$$

Formally the very same definition as for Cohen real σ_c, but the result will be completely different. In the definition of σ_r there are only generators of $\mathbf{B}(\omega)$, while in σ_c it was elements of the dense subset of $\mathbf{C}(\omega)$.

4.48 LEMMA. *(i)* $\sigma_r, \rho_c \in V[G]$,

(ii) $\sigma_r : \omega \to \{0, 1\}$,

(iii) $\rho_r \subset \omega$ *is infinite and splitting set.*

Proof. *(i)* Similar as in a proof for Cohen reals.
(ii) For any $n \in \omega$ the set $\{\langle s\rangle : s \in {}^n\{0,1\}\}$ is a maximal antichain, so there is exactly one $s \in {}^n\{0,1\}$ such that $\langle s\rangle \in G$. Thus $\sigma_r : \omega \to \{0,1\}$.
(iii) Let $Y \subseteq \omega$ be an infinite set. Assume that $Y \cap \rho_r \subset k$, for some $k \in \omega$. It means that for every finite $K \subset Y - k$, the clopen set $O_K = \{f \in \mathcal{C} : (\forall i \in K)\ f(i) = 0\}$ belongs to G, since there is $s \in Seq(2)$, $\langle s\rangle \in G$ and $K \subset dom(s)$, so $O_K \supset \langle s\rangle$. Now $m(O_K) = \frac{1}{2^{|K|}}$, thus $Z = \bigcap\{O_K : K \in [Y - k]^{<\omega}\} \in$ *Null*. This is a contradiction with set-completeness of G

We proved that $(\forall k \in \omega)$ there is $\langle s\rangle \in G$ so that there exists $i \in Y - k$ such that $s(i) = 1$.

Similarly we show that $(\forall k \in \omega)$ there is $\langle s\rangle \in G$ so that there exists $i \in Y - k$ such that $s(i) = 0$. It follows that ρ_r is a splitting real. $\square$

4.49 LEMMA. *Random forcing adds eventually different real.*

Proof. No bounded mapping $\sigma : \omega \to \omega$, $\sigma \in V[G]$ could be possibly eventually different, so $\sigma_r : \omega \to \{0,1\}$ even less. But from σ_r we can easily build eventually different real.

Take a partition $\{I_n : n \in \omega\}$ of ω into intervals of increasing length, e.g. $|I_n| = n$, for each $n \in \omega$. Consider sets $S_n = {}^{I_n}\{0,1\}$. Then $|S_n| = 2^n$ and let us fix an enumeration $S_n = \{s^n_i : i < 2^n\}$.

In $V[G]$ for each $n \in \omega$ there is some $i < 2^n$ so that $\sigma_r \restriction I_n = s^n_i$, since clearly the set $\{\langle s^n_i\rangle : i < 2^n\}$ is a maximal antichain in Random forcing.

If we put $\sigma(n) = i$ if $s^n_i = \sigma_r \restriction n$, then $\sigma \in \prod_{n\in\omega} 2^n$ is the desired real.

It is necessary and sufficient to check that for any $f \in \mathcal{C}$, $f \in V$ there is only finitely many n's such that $f \restriction I_n = \sigma_r \restriction I_n$. It follows from the properties of measure that

$$m(\bigcup_{n>k} \langle f \restriction I_n\rangle) \leq \sum_{n>k} \frac{1}{2^n} = \frac{1}{2^k},$$

hence $m(\bigcap_{k\in\omega} \bigcup_{n>k} \langle f \restriction I_n\rangle) = 0$. Since generic filter G is a set complete for countable families, that there is $m_0 \in \omega$ such that $\langle f \restriction I_n\rangle \notin G$ and $\sigma_r \restriction I_n \in G$, for each $n > m_0$. □

4.50 LEMMA. *Random forcing is ω^ω-bounding.*

This lemma follows from a general property that is common for large class of forcing notions.

4.51 DEFINITION. Weak distributivity. A forcing P is *weakly distributive*, or $(\omega, *, \omega)$-distributive if for any countable family $\{\mathcal{A}_n : n \in \omega\}$ of maximal antichains there is a dense subset $H \subseteq P$ such that each element of H is compatible with at most finitely many elements from each $\mathcal{A}_n$, i.e.

$$(\forall p \in H)\ (\forall n \in \omega) \text{ is the set } \{q \in \mathcal{A}_n : p \parallel q\} \text{ finite.}$$

It should be clear that P is weakly distributive iff $RO(P)$ is.

4.52 PROPOSITION. *A forcing notion P is ω^ω-bounding iff the algebra $RO(P)$ is (ω, ω, ω)-distributive; i.e. for any countable family $\{\mathcal{A}_n : n \in \omega\}$ of at most countable maximal antichains in $RO(P)$ there is a dense subset $H \subseteq RO(P)$ such that*

$$(\forall p \in H)\ (\forall n \in \omega) \textit{ is the set } \{q \in \mathcal{A}_n : p \parallel q\} \textit{ finite.}$$

We have to use (ω, ω, ω)-distributivity instead of weak distributivity here, because there are ω^ω-bounding forcigs without any countable maximal antichain, e.g. $([\omega]^\omega, \subseteq^*)$. Note that for ccc forcings these two notions of distributivity coincides.

INTERRELATIONSHIP OF $\mathcal{C}$ AND $\mathcal{N}$.

4.53 THEOREM. *(i) Random udela z groundmodel reals meager*

(ii) Cohen udela z groundmodel reals Null

4.54 HECHLER FORCING. Let $p \subset {}^{<\omega}\omega$ be ω-ary tree. A tree is called *Hechler tree* if

(i) it has a stem $s \in {}^{<\omega}\omega$ and

(ii) $(\forall t \in p)\ (s \subset t \to \mathrm{split}_p(t) = \{i \in \omega : t^\frown i \in p\})$ is a cofinal set.

Hechler forcing, denoted by H, consists of all Hechler trees ordered by inclusion.

4.55 FACT. $(\mathrm{H}, \subseteq)$ *is a separative partial ordering that is σ-centred, so* ccc.

Proof. Hechler trees with the common stem form a centred family and there is only finitely many different stems. □

It is quite frequent in literature, that by the Hechler forcing is meant the following

4.56 DEFINITION. *Hechler forcing* is a set $\mathrm{H}_0 = \{\langle s, f\rangle : s \in {}^{<\omega}\omega,\ f : \omega \to \omega,\ s \subset f\}$, with partial ordering $\langle s, f\rangle \leq \langle t, g\rangle$ if and only if $t \subseteq s\ \&\ (\forall n \in \omega)\ f(n) \geq g(n)$.

Forcings H and H_0 are different but they have very similar properties. By our opinion it is easier to work with H instead of H_0.

4.57 PROPOSITION. $(\mathrm{H}_0, \leq)$ *is a partial ordering that can be regularly embedded into* $(\mathrm{H}, \subseteq)$*, i.e.* $\mathrm{H}_0 \lessdot \mathrm{H}$.

Proof. Let $\langle s, f\rangle$ be a condition from H_0. Define ω-ary tree $p(s, f) \in \mathrm{H}$ such that $t \in p(s, f)$ iff $t \subseteq s$ or ($t \supseteq s$ and $(\forall i \in \mathrm{dom}(t))\ (i \geq |s| \to t(i) \geq f(i))$). $p(s, f)$ is Hechler tree with stem s and with uniformly determined splitting sets, i.e. $\mathrm{split}(t) = \omega - f(n)$ for each $t \in {}^n\omega$, $n \geq |s|$. It is easy to check that the mapping $\langle s, f\rangle \mapsto p(s, f)$ is isomorphism of $(\mathrm{H}_0, \leq)$ onto $(\{p(s, f)\}, \subseteq)$ and it is regular embedding. □

4.58 LEMMA. *Hechler forcing adds dominating reals and so it adds independent reals.*

Proof. Let G be a generic filter on H over V, then

$$\tau = \bigcup\{\mathrm{stem}(p) : p \in G\}$$

is a dominating real.Let $f : \omega \to \omega$ be a function from groundmodel. For $p \in \mathrm{H}$ consider $p' \subset p$, where

$$p' = \{\ s \in p\ :\ s \subset \mathrm{stem}(p) \text{ or } (\mathrm{stem}(p) \subset p\ \&\ (\forall i \geq |s|)\ p'(i) > f(i)\ )\}.$$

The set $\{p' : p \in \mathrm{H}\}$ is dense in H thus there is some $p' \in G$. We immediately get that $\tau(i) > f(i)$ for each $i \geq \mathrm{stem}(p')$.

If there is a dominating real in the extension, then there is also an independent real. It is enough to consider a partition of ω into intervals $I_0 = [0, a_0) \cup I_1 = [a_0, a_1) \cup I_2 = [a_1, a_2), \cup \ldots$ so that $X \cap I_k \cup I_{k+1} \neq \emptyset$. Then $\bigcup_k I_{4k} \cup I_{4k+1}$ is a splitting set. □

4.59 FACT. *Hechler forcing adds a Cohen real.*

Proof. We use Criterion 2, to the mapping $h : \mathrm{H} \to \mathbf{C}$, $h(p) : n \to \{0, 1\}$ and $h(i) = s(i)_{\mathrm{mod}(2)}$, for each $i < n = |s|$, where s is a stem of p. □

LUZIN SETS.

Interesting properties of Hechler forcing concerning Luzin sets.

4.60 DEFINITION. A set $L \subset \mathcal{C}$ is called *Luzin set* (or $(2^{\aleph_0}, \omega_1)$-Luzin set), if $|L| = 2^\omega$ and $|L \cap M| < \omega_1$ for each meagre set M.

We have to omit a proof of the following theorem.

4.61 THEOREM. *Let* G *be a generic filter on* H *over* V. *Then there is a Luzin set in the extension* V[G].

Now look at what happen to Cihon diagram (see **??**) by adding one Hechler real.

There are just two values for cardinal invariant in diagram, namely ω_1 and 2^ω and those values are independent on the values in the groundmodel.

> *Non*$(\mathcal{M}) = \omega_1$, since subset $X \subset L$ of Luzin set of size ω_1 cannot be meagre. On the other hand
>
> *cov*$(\mathcal{M}) = 2^\omega$, i.e. is of full size, since we have to also cover a Luzin set L and each meagre set covers only a countable part of L.

COMPLETE BOOLEAN ALGEBRA.

We focus on the description of the complete Boolean algebra RO(H) determined by the Hechler forcing H.

We work in the space $\mathcal{N}$. For $p \in H$, $[p]$ is the set of all branches of tree p and it is closed subset of $\mathcal{N}$. Such sets satisfy the crucial property to form a topology base:

$$(\forall f \in [p_1] \cap [p_2])\ (\exists q \in H)\ f \in [q] \subset [p_1] \cap [p_2].$$

We denote τ_H the topology on ${}^\omega\omega$ given by this open base. Topology τ_H extends standard topology of $\mathcal{N}$.

4.62 CLAIM. *The topological space* $({}^\omega\omega, \tau_H)$ *satisfies the Baire category theorem, i.e. any nonempty open set is not meagre.*

Proof. We show that base set $[p]$ cannot be covered by meagre set, i.e. the set $[p] \cap \bigcap_{n\in\omega} \bigcup\{[q] : q \in P_n\}$ is not empty for each countable collection of open dense sets $\{P_n : n \in \omega\}$ Let $p \in H$, let Q_n be arbitrary open dense (in the sense of ordering) set in H, for each $n \in \omega$; note that the set $\{[q] : q \in Q_n\}$ is dense in τ_H. Since Q_0 is dense, then there is a Hechler tree $q_0 \leq p$, $q_O \in Q_0$ and since Q_0 is open, one can assume, that $|\mathrm{stem} q_0| > 0$. We proceed by induction and for $n+1 \in \omega$ we fix $q_{n+1} \leq q_n$, $q_{n+1} \in Q_{n+1}$ and $|\mathrm{stem} q_{n+1}| > n+1$. The branch $f = \bigcup_n \mathrm{stem} q_n$ is in $[p] \cap \bigcap_{n\in\omega} \bigcup\{[q] : q \in Q_n\}$, which completes the proof. □

4.63 COROLLARY. *We immediately obtain that* $\mathrm{BOREL}({}^\omega\omega, \tau_H) = \mathrm{BOREL}(\mathcal{N})$ *so*

$$RO(H) \approx \mathrm{BOREL}(N)/\mathit{Meagre}(\tau_H) \approx BP(\tau_H)/\mathit{Meagre}(\tau_H).$$

4.64 ADDING PSEUDOINTERSECTION TO A FILTER.

4.65 THUEMMEL FILTER LAVER.

The following forcing notions does not satisfy ccc, in fact are $(2^\omega)^+$-cc. They collapse the cardinal 2^ω in groundmodel to smaller cardinal depending on the other properties of groundmodel, but the first uncountable cardinal $(\omega_1)^V$ is preserved by all of them.

4.66 MATHIAS FORCING. This ordering apart from being used to produce interesting generic extension, influenced a development of infinite Ramsey theory.

Conditions in Mathias forcing have the same structure as conditions in $P(F)$, see **??**. Instead of a filter F we consider all infinite subsets of ω.

STANDARD DEFINITION.

Mathias forcing consists of the sets

$$M = \{\langle s, A\rangle : s \in [\omega]^{<\omega},\ A \in [\omega]^\omega \ \&\max s < \min A\ \},\ \text{ordered by}$$

$\langle s, A\rangle \leq \langle t, B\rangle$ iff $s \supseteq t,\ A \subseteq B$ and $s - t \subseteq B$.

4.67 FACT. $(M, \subseteq)$ *is a separative partial ordering of size* 2^ω *with the largest element* $\langle\emptyset, \omega$*, satisfying* $(2^\omega)^+$-cc.

4.68 PROPOSITION. *Non separative partial ordering* $([\omega]^\omega, \subseteq)$ *can be regularly embedded into* $(M, \subseteq)$.

$$\begin{array}{rcl} ([\omega]^\omega, \subseteq) & \hookrightarrow & (M, \subseteq) \\ A & \longmapsto & \langle\emptyset, A\rangle. \end{array}$$

4.69 ISOMORPHIC REPRESENTATION OF MATHIAS FORCING AND INFINITE RAMSEY THEORY. Any condition $\langle s, A\rangle \in M$ determines so called *Mathias set*

$$E(s, A) = \{X \in [\omega]^\omega : s \sqsubset X \ \&\ X - s \subset A\}.$$

Notice that if $S \subsetneqq [\omega]^\omega$, then S is a Mathias set if

$s = \bigcap\{X : X \in S\}$ is finite and

$\{X - s : X \in S\} = [A]^\omega$ for some infinite set A.

4.70 FACT. $(M, \subseteq)$ *is isomorphic to* $\{\langle s, A\rangle : \langle s, A\rangle \in M\}$ *ordered by the usual inclusion.*

4.71 ELLENTUCK TOPOLOGY. When we speak about $\mathcal{P}(\omega)$ as a topological space, we usually mean the topology of Cantor discontinuum obtained via identification of subset of ω with their characteristic functions. $[\omega]^\omega \subseteq \mathcal{P}(\omega)$, hence we have an inherited topology and denote it topology τ_c. Notice that $([\omega]^\omega, \tau_c)$ is completely metrizable space, since $[\omega]^\omega$ is G_δ subset of $\mathcal{P}(\omega)$.

Ellentuck topology extends τ_c and its open base is formed by all Mathias sets. We denote this topologycal space by $\mathcal{E}$.

4.72 PROPOSITION. *(i)* $\mathcal{E}$ *is well defined topological space,*

(ii) $\mathrm{BOREL}(\mathcal{E}) = \mathrm{BOREL}([\omega]^\omega, \tau_c)$ *and*

(iii) $\mathcal{E}$ *satisfies the Baire category theorem.*

Proof. HINT.

(i) If $x \in S_1 \cap S_2$ for some Mathias sets S_1 and S_2 then there is a Mathias set S such that $x \in S \subseteq S_1 \cap S_2$.

(ii) Follows from the fact that any Mathias set is closed in τ_c and for each $t \in {}^n\{0,1\}$, $[t] \cap [\omega]^\omega$ is a Mathias set, namely $\langle t^{-1}\{1\}, \omega - n\rangle$.

(iii) similar argument as for the Hechler forcing, see **??**.

□

Now, it is easy to describe complete Boolean algebra that is determined by Mathias forcing. All the following algebras are isomorphic:

$$\mathrm{RO}(M) \simeq \mathrm{RO}(\mathcal{E}) \simeq \mathtt{Compl}\left(\mathrm{BOREL}(\mathcal{E})/\mathit{Meagre}(\mathcal{E})\right),$$

where $\mathrm{RO}(M)$ is complete Boolean algebra with dense set isomorphic to $(M, \leq)$ and $\mathrm{RO}(\mathcal{E})$ is the algebra of regular open sets of the space $\mathcal{E}$ ordered by $\subseteq$. Since the algebra $\mathrm{BOREL}(\mathcal{E})/\mathit{Meagre}(\mathcal{E})$ is only σ-complete, we have to take its completion here.

4.73 INFINITE RAMSEY THEORY. The well known classical Ramsey theorem stated in arrow notation

$$(\forall k > 0)\ \omega \longrightarrow (\omega)^k_2,$$

has a natural generalisation $\omega \to (\omega)^\omega_2$ that does not hold in ZFC. In other words, there is a colouring $c : [\omega]^\omega \to \{0,1\}$ such that for each infinite subset $A \subset \omega$ there are $A_1, A_2 \in [A]^\omega$ of different colour, $c(A_1) \neq c(A_2)$.

Each subset $S \subset [\omega]^\omega$ determines a colouring $c_S : [\omega]^\omega \to \{0,1\}$ so that $c(X) = 1$ iff $X \in S$.

4.74 DEFINITION. (i) $S \subset [\omega]^\omega$ is called *Ramsey* if for each infinite $A \subset \omega$ there is its infinite subset $B \in [A]^\omega$ so that either $[B]^\omega \subset S$, or $[B]^\omega \cap S = \emptyset$.

(ii) $S \subset [\omega]^\omega$ is called *completely Ramsey* if for each Mathias condition $\langle s, A\rangle$ there is $B \in [A]^\omega$ so that either $\langle\langle s, B\rangle\rangle \subset S$, or $\langle\langle s, B\rangle\rangle \cap S = \emptyset$.

We state here without proof.

4.75 THEOREM. (Ellentuck Theorem) *Subsets of the space $\mathcal{E}$ has the Baire property if and only if it is completely Ramsey.*

The following theorem and techniques used in its proof substantially influenced the development of infinite Ramsey theory. Now it is an easy corollary to more general Ellentuck theorem.

4.76 THEOREM. (Galvin - Prikry Theorem) *Each Borel set in classical topology τ_c is completely Ramsey.*

We later use this theorem in a proof of a Laver property of Mathias forcing.

4.77 HISTORICAL NOTE. For any topological space (X, τ) we can consider the space of its non-empty closed subsets $\mathtt{Closed}^+(\tau)$. Vietoris defined natural topology on this space, now known as Vietoris topology. Reader familiar with this concept may note that when starting from the discrete space ω the Vietoris topology is a topology on $\mathcal{P}(\omega)$ and its trace on $[\omega]^\omega$ is exactly the Ellentuck topology.

4.78 BACK TO MATHIAS FORCING. We will define an auxiliary sequence $\langle \leq_n : n \in \omega \rangle$ of orderings on M by $\langle s, A \rangle \leq_n \langle t, B \rangle$ if $s = t$, $A \subset B$ and first n elements of B are in A.

It is clear that

$$\leq \supseteq \leq_0 \supseteq \leq_1 \supseteq \ \ldots \ \supseteq \leq_n \supseteq \ \ldots$$

and if $\langle s, A \rangle \leq_n \langle t, B \rangle$ then both conditions have the same stem.

A sequence of conditions $\langle s, A_0 \rangle \geq_1 \langle s, A_1 \rangle \geq_2 \cdots \geq_n \langle s, A_n \rangle \geq_{n+1} \ldots$ is called a *fussion sequence* for M and $p = \langle s, \bigcap_i A_i \rangle$ is again a Mathias condition and $p \leq_n \langle s, A_n \rangle$, for each $n \in \omega$.

Our aim is to show that Mathias forcing with orderings $\leq_n$ $(n \in \omega)$ satisfies the following general principle introduced by J. Baumgartner. At first sight it seems little odd, but it appears to be useful.

4.79 DEFINITION. Axiom A. A forcing notion $(P, \leq)$ satisfies *axiom A* if there is a sequence $\langle \leq_n, n \in \omega \rangle$ of orderings of P such that

(i) $\leq \supseteq \leq_0$ and $\leq_n \supseteq \leq_{n+1}$, i.e. $p \leq_{n+1} q$ implies that $p \leq_n q$ and also $p \leq q$,

(ii) if $\langle p_n : n \in \omega \rangle$ is a sequence of conditions such that $p_{n+1} \leq_n p_n$ then there is $p \in P$ such that $p \leq_n p + n$ for each $n \in \omega$,

(iii) for any $p \in P$, any maximal antichain A and $n \in \omega$ there is $q \leq_n p$ and q is compatible with at most countably many $a \in A$; $|\{a \in A : a || q\}| \leq \omega$.

4.80 EXAMPLE. If forcing notion $(P, \leq)$ is ccc or σ-closed then it satisfies the axiom A. It suffices to put $\leq_n = \leq$ for each $n \in \omega$. For ccc forcing, there is nothing to check.

If P is σ-closed then property *(ii)* follows from 'σ-closedness'.

4.81 PROPOSITION. *Mathias forcing satisfies axiom A, with respect to orderings* $\{\leq_n : n \in \omega\}$*, see* *4.84.*

Proof. Fussion argument guarantees the property *(ii)* of axiom A.

To show *(iii)*, let $p = \langle s, A \rangle$, $n \in \omega$ and $\mathcal{A}$ be given antichain. Let K_0 be the set of first n-elements of A and $\{s_i : i < 2^n\}$ be an enumeration of all subsets of K_0, we may suppose that $s_0 = \emptyset$.

By recursion on $i < 2^n$ we construct non-increasing sequence $\langle S_i : i \in \omega \rangle$ of infinite subsets of A and simultaneously non-decreasing sequence $\langle \mathcal{A}'_i : i < 2^n \rangle$ of finite subsets of $\mathcal{A}$.

We start with condition $\langle s, S \rangle$ where $S = A - K_0$, note that $s \cup s_0 = s$.

Either there is a couple $r \in \mathcal{A}$ and $S_0 \in [S]^\omega$ such that $\langle s, S_0 \rangle$ is below r. If so, choose one, say r_0, S_0 and set $\mathcal{A}'_0 = \{r_0\}$. So we have the set S_0.

Or such couple does not exist. Then put $\mathcal{A}'_0 = \emptyset$ and $S_0 = S$. Assume we know S_k, $\mathcal{A}'_k$ and $k + 1 < 2^n$. Now we test condition $\langle s \cup s_{k+1}, S_k \rangle$. Again either we can choose a couple r_{k+1}, $S_{k+1} \in [S_k]^\omega$ that fulfils $\langle s \cup s_{k+1}, S_{k+1} \rangle \leq r_{k+1}$. Then we have S_{k+1} and $\mathcal{A}'_{k+1} = \mathcal{A}'_k \cup \{r_{k+1}\}$. Or no such couple exists, then $S_{k+1} = S_k$ and $\mathcal{A}'_{k+1} = \mathcal{A}'_k$.

When $k = 2^{n-1}$, first step is finished and we denote $A_1 = S_{2^{n-1}}$ and $\mathcal{A}_1 = \mathcal{A}'_{2^{n-1}}$. Notice that $|\mathcal{A}'_1| \leq 2^n$.

So $p_1 = \langle s, K_0 \cup A_1 \rangle \leq_n \langle s, A \rangle$, but we do not know that p_1 is compatible with only countably many conditions from $\mathcal{A}$.

We have to continue other ω-many steps. Take a set K_1 consisting of first $n + 1$ elements of $\mathcal{A}_1$ and enumerate $\{s_i : i < 2^{2n+1}\}$ of subsets of $K_0 \cup K_1$. We continue with the same process, starting with condition $\langle s, S \rangle$, where $S = A_1 - k_1$. By recursion choose appropriate couples r, S_i, i.e. $\langle s \cup s_i, S_i \rangle \leq r$ provided it is possible and enrich the starting set $\mathcal{A}_1$. Note that it is sufficient to consider s_i for which $s_i \cap K_1 \neq \emptyset$, other cases are covered by $\mathcal{A}_1$.

We obtain a finite $\mathcal{A}_2 \supseteq \mathcal{A}_1$, and $A_2 = S_{2^{2n}}$. Take K_2 first $n+1$ elements of A_2 and continue up to ω.

We get pairwise disjoint sets K_j $(j \in \omega)$, with $|K_0| = n$, $|K_j| = n+1$, for $j > 0$, and each K_j consists of first $n+1$ elements of A_j for $j > 0$.

It remains to put $q = \langle s, \bigcup_j K_j \rangle$ and it is clear that $q \leq_n p$. We need to show that q is compatible with at most countably many elements of $\mathcal{A}$. To this end, let us show that for $r \in \mathcal{A}$, $r \notin \bigcup_j \mathcal{A}_j$, $q \perp r$.

For if $q \| r$, then there is some t, $t \leq q$, $t \leq r$. $t = \langle s, B \rangle$, with finite L, so $s - L \subseteq \bigcup_{j<m_0} K_j$ for some $m_0 \in \omega$. But our construction guarantees that L is compatible with some condition from $\mathcal{A}_{m_0}$, which contradicts the disjointness of $\mathcal{A}$. □

The following assertion holds true for any forcing notion satisfying axiom A, the only specific but important for Mathias frocing is the fact that stems are preserved.

4.82 THEOREM. *Let $\{\mathcal{A}_n : n \in \omega\}$ be a countable family of maximal antichains in M. Then for any condition $p = \langle s, A \rangle$ there is a condition $q = \langle s, B \rangle$ such that*

(i) $q \leq p$ *and*

(ii) q *is compatible with at most countably many elements of $\mathcal{A}_n$, for each $n \in \omega$.*

Proof. □

Mathias forcing has moreover a property, called *Laver property*, that is not generally a consequence of axiom A.

4.83 PROPOSITION. *Let B be a complete Boolean algebra in which Mathias ordering $(M, \leq)$ is dense, i.e. $B \approx RO(M)$. Let $\{\mathcal{A}_n : n \in \omega\}$ be a countable family of maximal antichains in B. Then for any condition $p = \langle s, A \rangle$ there is a stronger condition $q = \langle s, B \rangle$ with the same stem such that q is compatible with at most 2^n elements of $\mathcal{A}_n$, for each $n \in \omega$. Particulary for any element $b \in B$ and any condition $\langle s, A \rangle$ there is a condition $\langle s, B \rangle \leq \langle s, A \rangle$ such that either $\langle s, B \rangle \leq b$ or $\langle s, B \rangle \perp b$.*

Proof. □

4.84 DIFFERENT FORMS OF DISTRIBUTIVITY LAWS. Let $(P, \leq)$ be a forcing notion and B be a complete Boolean algebra, $B = RP(P)$.

If $\{\mathcal{A}_i : i < n\}$ is a finitely many maximal antichains then there is a maximal antichain $\mathcal{A}$ refining all $\mathcal{A}_i$'s, i.e. for every $p \in \mathcal{A}$ and for each $i < n$ there is $q \in \mathcal{A}_i$ so that $p \leq q$.

The common refinement for infinitely many antichains need not exist.

4.85 DEFINITION. Let κ be an infinite cardinal. An ordering $(P, \leq)$ is called κ-*distributive* if there is a common refinement for each collection of κ many maximal antichains.

Non distributivity number, denoted as $\mathfrak{h}(P)$ is the minimal cardinal such that $(P, \leq)$ is not κ-distributive.

4.86 FACT. *(i)* $(P, \leq)$ *is κ-distributive if and only if an intersection of at most κ many open dense subsets is again an open dense subset.*

(ii) B *is κ distributive iff some dense subset $(P, \leq)$ is κ distributive.*

(iii) Non distributivity number is well defined provided that $(P, \leq)$ *has no atoms, in this case* $\mathfrak{h}(P) = \mathfrak{h}(RO(P))$ *and it is regular cardinal.*

(iv) If $(P, \leq)$ *is atomless and homogeneous, i.e.* $(P, \leq)$ *and* $((\mathtt{leftarrow}, p], \leq)$ *have isomorphic dense sets, then* $\mathfrak{h}(P) = \min\{\kappa : (\exists \{\mathcal{A}_i : i < n\})$ *maximal antichains such that for a dense subset* $H \subseteq P$ $(\forall p \in H)$ $(\exists \alpha < \kappa)$ $(\exists q_1, q_2 \in \mathcal{A}_\alpha)$ $q_1 \neq q_2$ & $p \parallel q_1$ & $p \parallel q_2 \}$.

4.87 DEFINITION. Three parametric distributivity.

(i) Let κ be infinite, $\lambda, \mu \geq 2$ cardinals. We say that a complete Boolean algebra B is (κ, λ, μ)-*distributive* if for any κ many maximal antichains $\{\mathcal{A}_\alpha : \alpha < \kappa\}$ each of size at most λ there is a dense set H such that for any $p \in H$ and every $\alpha < \kappa$ p is compatible with less then μ elements of each $\mathcal{A}_\alpha$.

(ii) $(\kappa, \cdot, \mu)$ distributivity means that the antichains are not limited in size, i.e. $(\kappa, \cdot, 2)$-distributivity is κ-distributivity.

(iii) A Boolean algebra B is *weakly distributive* if it is $(\omega, \cdot, \omega)$-distributive.

(iv) A forcing $(P, \leq)$ is called ${}^\omega\omega$*-bounding* if the algebra $RO(P)$ is (ω, ω, ω)-distributive.

Some of these notions we already encountered in Random forcing section. Measure algebra is weakly distributive and Random forcing is ${}^\omega\omega$-bounding.

4.88 DEFINITION. Let B be a complete Boolean algebra.

(i) B has *Laver property* if for any countable family $\{\mathcal{A}_n : n \in \omega\}$ of finite maximal antichains there is a dense subset $H \subseteq B^+$ such that

$$(*) \qquad (\forall p \in H)\ (\forall n \in \omega) \quad |\{\, q \in \mathcal{A}_n \ : \ q \parallel p \,\}| \ \leq\ 2^n.$$

(ii) B has *Sacks property* if for any countable family $\{\mathcal{A}_n : n \in \omega\}$ of countable maximal antichains there is a dense subset $H \subseteq B^+$ such that $(*)$.

Sack property is clearly stronger then Laver property, since maximal antichains need not to be finite, but at most countable.

INFLUENCE OF DISTRIBUTIVITY PROPERTIES OF FORCING TO EXTENSION.

4.89 PROPOSITION. *A forcing* $(P, \leq)$ *is* κ*-distributive if and only if in any generic extension* $V[G]$ *via* P *any mapping of form* $\sigma : \kappa \to V$, $\sigma \in V[G]$ *is a set in groundmodel; i.e.* $\sigma \in V$.

4.90 FACT. *Every forcing notion satisfying axiom A is* $(\omega, \cdot, \omega)$*-distributive.*

4.91 PROPOSITION. *If forcing* $(P, \leq)$ *satisfies* $(\omega, \cdot, \omega_1)$*-distributivity, then in any generic extension* $V[G]$ *via* P *for any mapping* $\sigma \in V[G]$, $\sigma : \omega \to V$ *there is a relation* $S \in V$ *such that* $S \subseteq \omega \times V$ *and* $|S(n)| \leq \omega$ *for each* $n \in \omega$ *and* $\sigma(n) \in S(n)$.

So P *preserves cardinal* ω_1.

4.92 PROPOSITION. *If forcing* $(P, \leq)$ *has a Laver property then in any generic extension* $V[G]$ *via* P *the following hold: Let* $\sigma : \omega \to \omega$, $\sigma \in V[G]$, *if there is* $f \in V$ *such that* $f : \omega \to \omega$ *and* $\sigma \leq f$ *then there is* $S : \omega \to [\omega]^{<\omega}$, $S \in V$ *such that* $(\forall n \in \omega)$ $\sigma(n) \in S(n)$ *and* $|S(n)| \leq 2^n$.

The previous proposition holds for Sacks forcing in a bit stronger way, to enclose the function f from the extension into a relation S we do not need to check that f is be bounded by the groundmodel function.

Important consequence of Laver property.

4.93 PROPOSITION. *If forcing has the Laver property then it does not add neither Cohen nor Random reals.*

4.94 SUMMARY OF PROPERTIES OF MATHIAS FORCING. Mathias forcing

(o) is homogenous, $2^{\omega+} - cc$ partial ordering.

(i) adds a dominating real, so also adds an independet real.

(ii) has laver property and so does not add neither Cohen nor Random reals.

(iii) satisfies axiom A, so does not collapses ω_1.

(iv) collapses all cardinal κ $\mathfrak{h} < \kappa \leq 2^\omega$ onto $\mathfrak{h}$, provided $\mathfrak{h} < 2^\omega$ in V; i.e. there is a one-to-one mapping $\sigma : \mathfrak{h} \to (2^\omega)^V$ and either smaller cardinals then $\mathfrak{H}$ or strictly larger cardinals then 2^ω are preserved.

Proof. HINT. For any partition $\{I_n : n \in \omega\}$, to intervals $I_n = [a_n, a_{n+1})$ the set of conditions $\{\langle s, A\rangle : \forall^\infty n |I_n \cap s| \leq 1\}$ is dense in Mathias forcing. So increasing enumeration of Mathias real is a dominating real.

The fact that Mathias forcing collapses cardinal follows from that $(\omega, \subseteq^*) \lessdot M$ and that $(\omega, \subseteq^*)$ is a forcing for such collaps.

□

The following two forcings do not add new reals. This means, among others, that every Polish space in the groundmodel remains Polish in the extension.

4.95 $([\omega]^\omega, \subseteq)$. The ordering $([\omega]^\omega, \subseteq)$ and its separative modification $([\omega]^\omega, \subseteq^*)$ were mentioned in chapter **??**.

Relevant properties of $([\omega]^\omega, \subseteq^*)$.

(i) atomless, homogeneous, $(2^\omega)^+$-cc and σ-closed.

(ii) Cardinal number $\mathfrak{h}$ denotes the non-distributivity number of this forcing, it is one of basic cardinal characteristics of the continuum.

(iii) $\mathfrak{h}$ is the minimal cardinal $\kappa \in V$ so that this forcing adds a new subset of κ.

(iv) Generaly this forcing adds a selective ultrafilter on ω.

(v) It adds a mapping from $\mathfrak{h}$ onto $(2^\omega)^V$, so it collapses all cardinals in V that are less orequal to 2^ω and larger then $\mathfrak{h}$.

Complete Boolean algebra $RO([\omega]^\omega, \subseteq^*)$ is the completion of $\mathcal{P}(\omega)/fin$ and this is isomorphic to the algebra of regular open sets of Čech-Stone reminder $\beta\omega - \omega$.

Let us first recall the definitions of some types of ultrafilters on ω.

4.96 DEFINITION. A non principal ultrafilter on ω is

(i) *selective (Ramsey)* ultrafilter if for any partition R of ω either

  (a) $R \cap U \neq \emptyset$ or

  (b) there is $X \in U$ such that $(\forall r \in R)\ |r \cap X| \leq 1$, i.e. X is a selector of R.

(ii) P-*ultrafilter* if for any partition R of ω either $R \cap U \neq \emptyset$ or there is $X \in U$ such that $(\forall r\ inR)\ |X \cap r| < \omega$.

(iii) Q-*ultrafilter* if for any partition R of ω into finite pieces there is $X \in U$ which is a selector for R.

It is easy to see that an ultrafilter U is selective if and only if it is simultaneously P and Q ultrafilter.

It should be noted that the existence of any such ultrafilter is not provable in ZFC. Selective ultrafilters exist e.g. when MA (Martin axiom) holds.

Proof. (i) Clear, (ii), (iii) follows from definition.

(iv) Let U be a generic filter, U is a filter on ω containing all co-finite sets. Since in V[U] there is no subset of ω, there is no new partition of ω. Let R be a partition of ω. If R is finite, then $R \cap [\omega]^\omega$ is predense and so $R \cap U \neq \emptyset$, therefore U is ultrafilter.

If R is infinite then a set $R \cup \{X : (\forall r \in R)(|X \cap r| \leq 1)\}$ is predense, so U contains either a member of R or some of its selectors. □

We show that in $([\omega]^\omega, \subseteq^*)$ there is so called base tree. It is a particular case of more general forcing notions.

4.97 PROPOSITION. (Base tree.) *Let $(P, \leq)$ be an ordering that is atomless, homogeneous, σ-closed and $|P| \leq 2^\omega$. Then there is a family $\{\mathcal{A}_\alpha : \alpha < (P)\}$ of maximal antichains such that*

(i) if $\alpha < \beta$ then $\mathcal{A}_\alpha$ refines $\mathcal{A}_\beta$,

(ii) $(\forall x \in P)\ (\exists \alpha < \mathfrak{h}(P))\ (\exists y \in \mathcal{A}_\alpha)\ y \leq x$,

(iii) $\forall y \in \mathcal{A}_\alpha\ |\{z \in \mathcal{A}_{\alpha+1} : z \leq y\}| = 2^\omega$.

The structure $(\bigcup_{\alpha<\mathfrak{h}(P)} \mathcal{A}_\alpha, \leq)$ is a tree by the general definition in chapter **??**, and the reason why this structure is caled base tree is that the undelrying set $\bigcup_{\alpha<\mathfrak{h}(P)}$ is dense in $(P, \leq)$.

Proof. Since $(P, \leq)$ is an atomless and σ-closed ordering, its distributivity number $\kappa = \mathfrak{h}(P)$ is well defined and any family of maximal antichains of size strictly less then κ has a common refinement and moreover κ is uncountable regular cardinal, $\kappa \leq 2^\omega$ for $|P| \leq 2^\omega$.

Take a family $\{Q_\alpha : \alpha < \kappa\}$ of refining maximal antichains. From homogenity of P we can suppose that every $p \in P$ is compatible with at least two elements of some Q_α.

We claim

$$(\forall p \in P)\ (\exists \alpha < \kappa) \quad |\{\ q \in Q_\alpha\ :\ p \parallel q\ \}|\ =\ 2^\omega.$$

Let $p \in P$ be given. By recursion for nodes φ of binary tree ${}^\omega\{0,1\}$ we choose labels $\langle \alpha_\varphi, p_\varphi \rangle$, where $\alpha_\varphi < \kappa$ and $p_\varphi \in P$. Fro $\emptyset$, put $\alpha_\emptyset = 0$ and $p_\emptyset = p$. If we already know $\alpha_\varphi, p_\varphi$, then for $\varphi_0 = \varphi^\frown\{0\}$ and $\varphi_1 = \varphi^\frown\{1\}$ set α_{φ_i} the least $\alpha < \kappa$ such that there are $q_0, q_1 \in Q_\alpha$, $q_0 \neq q_1$ such that $p_\varphi \parallel q_0$ and $p_\varphi \parallel q_1$ and choose p_{φ_0} as an element which is below p_φ and q_0 and p_{φ_1} below p_φ and q_1.

For arbitrary branch $f : \omega \to \{0,1\}$ we have descending chain $\{p_{f\restriction n} : n \in \omega\}$, since P is σ-closed there is an element p_f which is below all $p_{f\restriction n}$.

The set $\{p_f : f \in {}^\omega\{0,1\}\ \}$ has size 2^ω is an antichain, all p_f's are below p. Denote $\beta(p) = \sup\{\alpha_\varphi : \varphi \in {}^{<\omega}\{0,1\}\}$. Then $\beta < \kappa$ and p is compatible with 2^ω many elements of Q_β, since p_f, p_g are compatible with different elements of Q_β provided $f \neq g$.

The following is an easy corollary of the claim:

(a) for any $p \in P$ there is a maximal antichain in $(\leftarrow, p]$ of size 2^ω,

(b) there is an antichain $\mathcal{X}_\beta$ such that $(\forall p \in P)\ (\beta_p = \beta) \to (\exists x \in \mathcal{X}_\beta)\ x \leq p$.

For $p \in P$ pick up a maximal antichain $\mathcal{Y}(p)$ in $(\leftarrow, p]$ of size 2^ω. By recursion, define a base tree: Put $\mathcal{A}_0 = Q_0$, for limit ordinal α let $\mathcal{A}_\alpha$ be a common refinement of $\{\mathcal{A}_\beta : \beta < \alpha\}$ and let $\mathcal{A}_{\alpha+1}$ be a maximal antichain refining $\bigcup\{\mathcal{Y}_p : p \in \mathcal{A}_\alpha\}$, Q_α and $\mathcal{X}_\alpha$ □

4.98 COROLLARY. *Suppose that* $(P, \leq)$ *and* $(Q, \leq)$ *satisfy conditions of previous proposition and* $\mathfrak{h}(P) = \mathfrak{h}(Q) = \omega_1$. *Then* P *and* Q *have dense isomorphic subsets.*

Proof. HINT. Let $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ and $\{\mathcal{B}_\alpha : \alpha < \omega_1\}$ be corresponding base trees, we can assume that $|\mathcal{A}_0| = |\mathcal{B}_0| = 2^\omega$. Let φ_0 be one-to-one mapping of $\mathcal{A}_0$ onto $\mathcal{B}_0$, φ_0 can be extended to isomorphism of $(\bigcup\{\mathcal{A}_\alpha : \alpha \in \omega_1\}, \leq_1)$ onto $(\bigcup\{\mathcal{B}_\alpha : \alpha \in \omega_1\}, \leq_2)$. □

Another corollary of base tree Proposition 4.97 is the following lemma concerning forcing notion $(P, \leq)$ satisfying cc-properties (**??**). The particular case of this lemma proves remaining claims concerning $([\omega]^\omega \subseteq^*)$

4.99 LEMMA. *Let* $\kappa = \mathfrak{h}(P)$. *Then*

(i) P *does not add new subset of* λ *for* $\lambda < \kappa$ *and*

(ii) P *adds a mapping of* κ *onto* $(2^\omega)^V$, *so if* $\kappa < (2^\omega)^V$ *then* P *collapses groundmodel continuum onto* κ *in any generic extension via* P.

Proof. *(i)* for any $\lambda < \kappa$, $(P, \leq)$ is λ-distributive, see **??**.
(ii) Let $\{\mathcal{A}_\alpha : \alpha < \kappa\}$ be a base tree of P. For given $\alpha < \kappa$ choose a mapping $f_\alpha : \mathcal{A}_{\alpha+1} \to 2^\omega$ such that for every $a \in \mathcal{A}_\alpha$ $f_\alpha \restriction \{b \leq a\}$ is one-to-one mapping of the set $\{b \in \mathcal{A}_{\alpha+1} : b < a\}$ onto 2^ω. if G is a generic filter on P over V then $\tau = \{\langle \alpha, p\rangle : \alpha < \kappa,\ p < 2^\omega;\ (\exists a \in \mathcal{A}_{\alpha+1} \cap G)\ f_\alpha(a) = p\}$. is the desired mapping, that belongs to $V[G]$. □

4.100 CARDINAL NUMBERS $\mathfrak{h}_n$, FOR $n \geq 2$. Let $N \in \{n \in \omega : n \geq 2\} \cup \{\omega\}$, then the product $\prod_{n\in N}([\omega]^\omega, \subseteq^*)$ also satisfy cc-property. The distributivity number of such product is denoted $\mathfrak{h}_N$.

4.101 LEMMA. *Let* P, Q *are orderings satisfying* cc*-properties and there is a regular embedding of* P *into* Q. *Then* $\mathfrak{h}(Q) \leq \mathfrak{h}(P)$.

Proof. We proceed toward a contradiction. Suppose $\mathfrak{h}(P) < \mathfrak{h}(Q)$ and we can assume that $P \subseteq Q$.
Let $\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}(P)\}$ be a base matrix for P. Then $\mathcal{A}_\alpha$ are also maximal antichains in Q. Since $\mathfrak{h}(P) < \mathfrak{h}(Q)$, there is maximal antichain $\mathcal{A}$ in Q refining all $\mathcal{A}_\alpha$'s. Since the embedding of P into Q is regular, the complete Bolean algebra $RO(P)$ is a complete subalgebra of complete Boolean algebra $RO(Q)$. Let $S = \{upw(a) : a \in \mathcal{A}\}$, then $S \subset RO(P)^+$ and no element of P is below any element of S which is a contradiction since P is dense in $RO(P)$. □

The preceding lemma and the fact that the caninical embedding of $\prod_{i<n}([\omega]^\omega, \subseteq^*)$ into $\prod_{i<m}([\omega]^\omega, \subseteq^*)$ for $n \leq m$ is regular give us the basic inequality

$$\mathfrak{h} \geq \mathfrak{h}_2 \geq \mathfrak{h}_3 \geq \cdots \geq \mathfrak{h}_\omega.$$

In this generaly infinite sequence there are clearly only finitely many jumps; i.e. only finitely many inequalities can be strict. It is still an open problem if the jumps can ocur on arbitrarily prescribed places. It is known (Shelah and Spinas) that it is consistent that $\omega_2 = \mathfrak{h} > \mathfrak{h}_2 = \omega_1$.

4.102 CONTINUUM HYPOTHESIS. For every model of ZFC there is a generic extension that satisfy the continuum hypothesis and moreover the prediction principle $\diamond$ holds true.

Standard forcing notions for producing extension that satisfy CH are the following

$$(\mathrm{Fn}(\omega_1, \{0,1\};\ \omega_1),\ \supseteq),$$
$$(\mathrm{Fn}(\omega_1, \mathbb{R};\ \omega_1),\ \supseteq),$$
$$(\mathrm{Fn}(2^\omega, \{0,1\};\ \omega_1),\ \supseteq),$$

those forcings are equivalent, separatice partial orderings, satisfying cc-condition with distributivity number $\mathfrak{h}(P) = \omega_1$.

Let G be a geric filter on such ordering. By the previous it is clear that there are no new reals in V[G] and that there is is one-to-one mapping τ of ω_1 on $(2^\omega)^V$. Hence, if $\{r_\alpha : \alpha < (2^\omega)^V\}$ is the numbering of all subsets of ω in V then $\{r_{\tau(\alpha)} : \alpha < (2^\omega)^V\}$ is the ordering of $\mathcal{P}(\omega)$ in V[G]. So $V[G] \models$ CH.

4.103 DEFINITION. $\diamond$-*sequence*. A sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ is called $\diamond$-sequence if

(i) $A_\alpha \subset \alpha$ for every $\alpha \in \omega_1$ and

(ii) for any $X \subseteq \omega_1$ the set $\{\alpha \in \omega_1 : X \cap \alpha = A_\alpha\}$ is stationary in ω_1.

Gussing principle $\diamond$ that postulates the existence of a $\diamond$-sequence is due to R. Jensen. It is easy to see, that $\diamond$ implies CH.

Let us mention two well known consequences of $\diamond$ without proofs:

(a) there is a Souslin tree, so the negation of Souslin hypothesis holds true.

(b) (A. Ostaszewski) There is an Ostaszewski space; i.e. a topological space which is completely regular, hereditarily separable, countable compact, perfectly normal but non-compact space.