3. Forcing notion and Generic filters

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Bohuslav Balcar, balcar@math.cas.cz[1]
Tomáš Pazák, pazak@math.cas.cz[1]
Jonathan Verner, jonathan.verner@matfyz.cz[2]

We now come to the important definition.

3.1 Definition. Forcing Notion. A *forcing notion* is a synonym for an ordering (see Definition 2.1). In this context elements of a forcing notion P are called conditions and for two conditions $p, q \in P$ we say that p is *stronger* than q if $p \leq q$. If two conditions are compatible we write $p \parallel q$ and we write $p \perp q$ if they are incompatible or disjoint.

Note. Beware that some authors use the so called 'eastern notation' and they say that a condition p is stronger, i.e. carries more information, than q if $p \geq q$. When reading a book or article on forcing, one should check which convention is used.

Recall that for a forcing notion $(P, \leq)$ we have defined its separative modification $(P, \leq_{sp})$ which gives rise to the separative quotient $(P, \leq_{sp})/\approx$ and this in turn is isomorphic to a dense subset of a complete Boolean algebra we have denoted by $RO(P)$. Note that when dealing with properties which are interesting from the forcing point of view, it is usually not important whether we take the forcing notion, its separative modification or the complete Boolean algebra $RO(P)$. The reason lies in the preservation of the disjointness relation.

Since it is formally irrelevant whether one works with an ordering or a complete Boolean Algebra one may wonder whether there is really *any* difference. It turns out that orderings are easier to work with in practical applications, while complete Boolean algebras are useful for the theoretical work.

3.2 Comparing forcing notions.

3.3 Definition. Given two forcing notions P, Q we say that

(i) $P \lessdot Q$ if $RO(P)$ can be completely embedded into $RO(Q)$,

(ii) the orderings P and Q are *forcing equivalent*, $P \sim Q$ if the algebras $RO(P)$ and $RO(Q)$ are isomorphic.

The $\lessdot$ relation is transitive and $\sim$ is an equivalence relation. A special case of the $\sim$ relation is the case when the two forcing notions have isomorphic dense subsets. However note that $P \lessdot Q$ and $Q \lessdot P$ *does not*, in general, imply $P \sim Q$. We also have to be prepared that sometimes it is not so easy to determine whether two forcing notions are in these relations.

We introduce two criteria for $P \lessdot Q$.

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3.4 CRITERION 1. Let P, Q be two forcing notions and $f : P \to Q$ be a regular mapping, i.e.

(i) f is homomorphism of orderings that

(ii) preserves disjointness relation and

(iii) each maximal antichain A in P is mapped onto a maximal antichain in Q,

then $P \lessdot Q$.

Proof. (HINT) The mapping f preserves disjointness. Let $B = \mathrm{RO}(Q)$. Then there is a homomorphism h of $(P, \leq_1)$ into $(B^+, \leq)$ preserving disjointness and such that $\bigvee_B h[P] = \mathbf{1}$. The complete subalgebra of B completely generated by $h[P]$ is $\mathrm{RO}(P)$. □

3.5 EXAMPLE. (i) Let $P \subset (Q, \leq)$ be such that every maximal antichain in $(P, \leq)$ is maximal antichain in Q. Then $P \lessdot Q$, and the identity on P is a regular embedding.

(ii) Let P, Q be forcing notions with greatest elements $\mathbf{1}_P$ and $\mathbf{1}_Q$, respectively. Then $P \times \{\mathbf{1}_Q\}$ and $\{\mathbf{1}_P\} \times Q$ are regular suborderings of $P \times Q$ (see Definition 2.26).

3.6 CRITERION 2. Let $h : Q \to P$ satisfy

(i) h is a homomorphism,

(ii) $(\forall q \in Q)(\forall p' \leq h(q))(\exists q')(q' \parallel q \ \&\ h(q') \leq p')$,

(iii) and $h[Q]$ is dense in P.

Then $P \lessdot Q$.

Proof. (HINT) We can assume that Q is dense in $B = \mathrm{RO}(Q)$ and P is dense $C = \mathrm{RO}(P)$. Let $f : C \to B$, where $f(c) = \bigvee_B \{q \in Q : h(q) \leq c\}$. Then f is a complete embedding of C into B. □

3.7 PROJECTIONS. Let C be a complete subalgebra of a cBA B. The *upward projection* $\mathrm{upr} : B \to C$ is an onto mapping defined by $\mathrm{upr}(b) = \bigwedge_C \{c \in C : c \geq b\}$, for each $b \in B$. The element $\mathrm{upr}(b)$ is the least element in C which is larger than b. For any $b_1, b_2 \in B$ $\mathrm{upr}(b_1 \vee b_2) = \mathrm{upr}(b_1) \vee \mathrm{upr}(b_2)$. On the other hand we only have $\mathrm{upr}(b_1 \wedge b_2) \leq \mathrm{upr}(b_1) \wedge \mathrm{upr}(b_2)$.

The mapping $\mathrm{upr} : B^+ \to C^+$ satisfies all conditions for h in criterion 2.

3.8 THE GENERIC FILTER. Recall that a filter F on an ordered set P is an upward closed, centred subset of P (see Definition 2.2). The following notion of a *generic filter* is the key to understanding the whole theory of forcing. It is also probably the most confusing definition for beginners.

3.9 DEFINITION. Generic filter. A filter G on a forcing notion P is called a *generic filter* if it satisfies a condition of 'genericity':

$$(\forall D \subseteq P,\ D \text{ dense})(\exists d \in D)(d \in G).$$

The following are equivalent characterisations of genericity and follow from the Fact 2.3.

3.10 FACT. *For a filter G on P the following conditions are equivalent.*

(i) G *is generic,*

(ii) *whenever* $X \subseteq P$ *is predense* $G \cap X \neq \emptyset$*,*

(iii) *whenever* $A \subseteq P$ *is a maximal antichain* $G \cap A \neq \emptyset$*.*

Remember that when we talk about F being a filter on a Boolean algebra B we in fact mean that F is a filter on the canonical ordering $(B^+, \leq)$, in particular F does not contain $\mathbf{0}_B$.

3.11 FACT. *Let* B *be a cBA,* G *a filter on* B*. Then* G *is a generic filter on* B *if and only if*

(i) for any $b \in B^+$ *either* $b \in G$ *or* $-b \in G$*, i.e.* G *has the* ultrafilter property *and*

(ii) for any set $X \subset G$ *the meet* $\bigwedge_B X \in G$*, i.e.* G *is* set-complete.

Proof. For any $b \in B^+$, the family $\{b, -b\}$ is predense in B and so is the family $\{\bigwedge X\} \cup \bigcup\{\{x\}^\perp : x \in X\}$.

For the opposite direction let $A \subset B^+$ be a maximal antichain. If $A \cap G = \emptyset$, then $(\forall a \in A)$ $-a \in G$ and thus $\mathbf{0} = \bigwedge\{-a : a \in A\} \in G$, a contradiction. □

3.12 COROLLARY. *If* C *is a complete subalgebra of a cBA* B *and* G *is a generic filter on* B*, then* $G \cap C$ *is a generic filter on* C*.*

3.13 COMMENT. Let G be a generic filter on $(P, \leq)$. Then G is $\approx_{sp}$ saturated, i.e. if $x \in G$ and $x \approx_{sp} y$ then $y \in G$. This follows from the fact that $x \approx_{sp} y \rightarrow \{x\}^\perp = \{y\}^\perp \rightarrow \{y\} \cup \{x\}^\perp$ is predense in $(P, \leq)$. Thus G is also a generic filter on $(P, \leq_{sp})$.

Consider now the separative quotient $(P, \leq_{sp})/\approx$ and the relation $r_1 = \{\langle x, [x]_{\approx_{sp}}\rangle : x \in P\}$. Then $r_1[G] = G_1$ is a generic filter on the separative quotient and moreover $G = r_1^{-1}{}''G_1$. Moreover, considering $B = RO(P)$, with a homomorphism $h : P \rightarrow B$ on a dense set, let $r_2 \subseteq P \times B$ be a relation defined by $\langle p, a\rangle \in r_2 \equiv a \geq h(p)$. Then $r_2[G] = G_2$ is a generic filter on the cBA B and $G = r_2^{-1}{}''G_2$.

We can see that a generic filter on P determines, simply via images by set relations, a generic filter on any ordering Q, for which $Q \lessdot P$ (see Definition 3.3).

3.14 PROBLEM. It follows directly from Fact 3.11 that a generic filter needs to extend an atom to be 'set-complete'. In other words there are no non-trivial generic filters. Moreover all ordered sets we discussed so far are atomless and so there are *no* generic filters on them.

3.15 WHAT NOW? We need to break out of our universe of sets and admit semisets into our reasoning. It is not impossible, that a generic filter might exist as a semiset.

3.16 AN ANALOGY. Going back in time, imagine, that there are no numbers beyond real numbers for us. However this means, that some simple quadratic equations have no solution. At this point, we break out of our universe of real numbers and accept a single imaginary object $\sqrt{-1}$. Then, starting with this object and our universe of numbers $\mathbb{R}$, we are able to build an extension of $\mathbb{R}$ — the field of complex numbers — and extend algebraic operations to this new field. What we get is an algebraically closed field, i.e. not only algebraic equations with real coefficients now have a solution but also equations with complex coefficients.

One can look at a generic filter as a kind of imaginary object, in our terminology, a proper semiset.

Let us broaden our minds and admit that our universe is not absolute, but that there are possible extensions in which some proper semisets of our universe V become sets. A generic filter is the gate to such extensions in a similar way as $\sqrt{-1}$ was to the the extension of $\mathbb{R}$.

An extension of a universe of course not only contains all sets from the original universal class V, but also admits an extension of the relation $\in$ to new born sets.

As the field of complex numbers is the smallest field that contains $\mathbb{R}$ and $\sqrt{-1}$, in what follows we will try to convince you that one can consider the smallest extension of our universe $(V, \in)$ that contains a $V-$generic filter G. Such an extension, usually called the generic extension, will satisfy the axioms of set theory and will be denoted by $V[G]$.

3.17 DEFINITION. Extension. A transitive universe $(M,\in)$ is called an extension of a universe $(V,\in)$ if, in M, all the axioms of ZF are satisfied, $M \supseteq V$ and both universes have the same ordinal numbers.

When talking about an extension, the starting universe is often referred to as the *ground model* and is denoted by $(V,\in)$. Note that the notion of an extension is more general than the notion of a generic extension.

3.18 WHY GENERIC FILTERS? The driving motive behind the definition of a generic filter is the quest for simplicity: we want the extension to "essentially depend on a single semiset". This, together with the natural requirement that the extension is closed under simple set operations, already gives us the generic filter. This paragraph will explain this in more detail.

Consider an extension $V \subseteq M$ of the ground model V as in 3.17. The class M is closed under standard binary operations, namely the Cartesian product $x \times y$ and the Boolean operations of union $x \cup y$ and set difference $x \setminus y$ and also intersection, since $x \cap y = x \setminus (x \setminus y)$. Moreover it is also closed under the operation $\operatorname{rng}(z) = \{x : (\exists x)(\langle x, y\rangle \in z\}$.

The following notation for semisets of M over V is convenient:

$$\mathrm{Sm}_{M,V}(\sigma) \equiv \sigma \in M \;\&\; (\exists a \in V)(\sigma \subseteq a).$$

When there is no danger of confusion, we drop the subscripts M, V. The basic properties of semisets of M over V are the following:

(i) $(\forall x \in V)(\mathrm{Sm}(x))$,

(ii) if $M \setminus V \neq \emptyset$ then there is a proper semiset

(iii) the class of semisets is closed under the Cartesian product, union and difference of sets and the range operation

(iv) and it is closed under taking images via ground model relations, i.e. if $\mathrm{Sm}(\sigma)$ and $r \in V$ then $\mathrm{Sm}(r[\sigma])$.

Only number (ii) needs comments: Using the axiom of foundation valid in M find an $\in$-minimal element σ of $M \setminus V$. The image of σ under the ground model rank function rk cannot be cofinal in On because $\mathrm{rk}[\sigma]$ is an element of M by the axiom of replacement and M and V have the same ordinals.

Note however, that not all elements of M are semisets over the ground model. For example the singleton $\{x\}$ consisting of some new set $x \in M \setminus V$ is *not* a semiset.

We now give a precise definition of dependence between semisets and also introduce the notion of their similarity.

3.19 DEFINITION. We say that σ is *dependent* on ρ, denoted $\mathrm{Dep}(\rho,\sigma)$, if there is a relation r from the ground model such that $\sigma = r[\rho]$. Two semisets σ, ρ are said to be *similar*, denoted $\mathrm{Sim}(\sigma,\rho)$, if both are dependent on the other.

Fix a nonempty semiset $\sigma \subseteq a$ of M over V. We can consider the class $\mathrm{Sm}_{\sigma,M,V}$ of semisets depending on σ:

3.20 FACT. *(i) Each ground model set depends on σ, i.e. $(\forall x \in V)(\mathrm{Dep}(\sigma, x))$ and*

(ii) if ρ_0, ρ_1 depend on σ then so does $\rho_0 \cup \rho_1$.

Proof. For (i) given any $x \in V$ take some $y \in \sigma$ and the relation $\{y\} \times x$ shows that x depends on σ. For two, given witnesses r_0, r_1 to the dependence of ρ_0, ρ_1 on σ, the relation $r_0 \cup r_1$ shows that $\rho_0 \cup \rho_1$ depends on σ. □

3.21 PROPOSITION. *The set* $\rho = \{b \subseteq a : b \in V,\ b \cap \sigma \neq \emptyset\}$ *depends on* σ.

Proof. This is witnessed by the relation $r = \{\langle x, b\rangle : x \in b, b \subseteq a, b \in V\}$. □

A natural question to ask is: When is the class $Sm_{\sigma,M,V}$ closed under set difference, i.e. given $\rho \in Sm_{\sigma,M,V}$ when is $\mathcal{P}(a) \setminus \rho$ also a member of $Sm_{\sigma,M,V}$?

3.22 PROPOSITION (Balcar,Vopěnka). *The semiset* $\mathcal{P}(a) \setminus \rho$ *depends on* σ *if and only if there is an ground model ordering* $\leq$ *on* a *such that* σ *is a generic filter on* $(a, \leq)$.

3.23 COROLLARY. *The class* $Sm_{\sigma,M,V}$ *is a ring iff* σ *is a generic filter on some ordered set from the ground model.*

Proof of proposition. Let $s \in V$ be a relation such that $s[\sigma] = \mathcal{P}(a) \setminus \rho = \{x \subseteq a : x \cap \sigma = \emptyset\}$. We can assume that $s \subseteq a \times \mathcal{P}(a)$. Define

$$\bar{s} = \{\langle x, y\rangle : x \in \mathrm{dom}(s)\ \&\ y \in \bigcup s[\{x\}]\}.$$

It is easy to see that

(i) if $x \in \sigma$ then $\bar{s}[\{x\}] \cap \sigma = \emptyset$ and

(ii) for any $b \subseteq a, b \in V$ if $b \cap \sigma = \emptyset$ then there is an $x \in \sigma$ such that $b \subseteq \bar{s}[\{x\}]$.

Let $d = (\bar{s} \cup \bar{s}^{-1}) \setminus id_a$. This is a symmetric and antireflexive relation belonging to V. Finally let

$$x \leq y \equiv d[\{x\}] \supseteq d[\{y\}], \quad \text{for } x, y \in a$$

Claim. The relation d has the following properties:

(iii) $(\forall x \in a)(x \in \sigma \equiv d[\{x\}] \cap \sigma = \emptyset)$ and

(iv) $\langle x, y\rangle \in d$ implies $x \perp y$ in the ordering $(a, \leq)$.

(iii) is straightforward to check and for (iv) suppose to the contrary that there is a $z \leq x, y$. Then $d[\{z\}] \supseteq d[\{x\}] \cup d[\{y\}]$ and $y \in d[\{x\}]$ so that $\langle z, z\rangle \in d$ a contradiction.

Claim. σ is a generic filter on $(a, \leq)$ over V. *Filter.* If $x \in \sigma$ and $y \geq x$ then $d[\{x\}] \supseteq d[\{y\}]$ so $d[\{y\}] \cap \sigma = \emptyset$ so by (iii) $y \in \sigma$. Next, given $x, y \in \sigma$ we know that $(d[\{x\}] \cup d[\{y\}]) \cap \sigma = \emptyset$, so, by (ii), there is $z \leq x, y$.

Genericity. Let $b \subseteq a$ be a dense set in $(a, \leq)$. If $b \cap \sigma \neq \emptyset$, then we are done. Otherwise, by (ii) and (iii), there is $z \in \sigma$ such that $d[\{z\}] \supseteq b$ and, by (iv), z is disjoint with all elements of b which contradicts the fact that b is dense.

The other implication in the proposition follows from the proof of the corollary. □

Proof of Corollary. It remains to check that the semisets depending on a generic filter σ on $(a, \leq)$ are closed under set difference and intersection. Let $\pi = r[\sigma] \subseteq b$ and $\tau = s[\sigma]$. Then

$$\pi \cap \tau = b \setminus ((b \setminus \pi) \cup (b \setminus \tau))$$

and

$$\pi \setminus \tau = b \setminus ((b \setminus \pi) \cup \tau).$$

It suffices to show that $b \setminus \pi$ depends on σ. But for this we can let

$$\bar{r} = \{\langle x, y\rangle : x \in (r^{-1}[\{y\}])^{\perp}, y \in b\}$$

and we have $\bar{r}[\sigma] = b \setminus \pi$: For $y \in b \setminus \pi$ we know that $r^{-1}[\{y\}] \cap \sigma = \emptyset$ so, since $r^{-1}[\{y\}] \cup (r^{-1}[\{y\}])^{\perp}$ is predense, there is $x \in \sigma$ such that $\langle x, y\rangle \in \bar{r}$. On the other hand if $y \in \pi$ then $r^{-1}[\{y\} \cap \sigma \neq \emptyset$ and no $x \in (r^{-1}[\{y\}])^{\perp}$ is a member of σ, hence $y \notin r[\sigma]$. □

Recall that in Comment 3.13 we used images of a generic filter to show that a generic on the order $(P, \leq)$ produces a generic on the complete Boolean algebra $RO(P)$. In fact those generic filters are similar. Moreover if $P \lessdot Q$ then given a generic G on Q the embedding naturally determines a filter $\bar{G}$ on P and $Dep(G, \bar{G})$ so $V[\bar{G}] \subseteq V[G]$ and $Sm_{V[\bar{G}],V} \subseteq Sm_{V[G],V}$.

3.24 SLOGAN. Since we will meet different universes of sets, we have to be more precise in using the term generic filter. We shall therefore say that

'G is generic filter on P over N'

or

'G is an N-generic filter on P'

to express that the ordering P is an element of N and the generic filter G intersects all dense subsets of P which *belong to* N (see Definition 3.9). Note that whenever P is atomless then $G \notin N$.

3.25 CONSISTENCY. The easiest way to see that adding a generic filter G to a ground model N produces a model of ZF (ZFC), or that G can be consistently added to N, is to pretend that N is a countable transitive model. Let P be an ordering, $P \in N$. Since N is countable there are only countably many dense subsets of P in N and we can use Rasiowa - Sikorski Theorem 2.4 to obtain a filter G that intersects all those dense sets. The smallest model $N[G]$ of ZF (ZFC) containing N and the set G is the generic extension of N and G is a generic filter on P over N.

3.26 HOW TO DESCRIBE A GENERIC EXTENSION $V[G]$? The theory of forcing does this via names that are available already in the ground model V. These names are then evaluated using the generic filter G as a parameter. We show one quick approach now, while in a subsequent chapter we describe the more standard approach.

Let $(P, \leq)$ be a forcing notion. In V we define a hierarchy of names. A name will be some relation $r \in V, \mathrm{dom}(r) \subseteq P$ together with a rank α. We denote the names of rank 1 as

$$R_1 = \{(r, 1) : \mathrm{dom}(r) \subseteq P\}$$

and inductively define

$$R_\alpha = \left\{(r, \alpha) : r \in R_1 \& \mathrm{rng}(r) \subseteq \bigcup_{\beta<\alpha} R_\beta\right\}.$$

Formally, names are pairs, but we shall say that r is a name of rank α instead of talking about a name (r, α).

Let G be V-generic filter G on P. The evaluation of a name r will depend on its rank. If r is of rank 1 then

$$r/_G = r[G].$$

Inductively, if r is of rank α, we let

$$r/_G = \{s/_G : s \in r[G]\}$$

Finally we let

$$V_\alpha[G] = \{r/_G : r \text{ is of rank } \alpha\} \; V[G] = \bigcup_{\alpha < On} V_\alpha[G]$$

The way $V[G]$ is built up is essentially by simulating the operation of power set. We shall illustrate this on a countable transitive model M and an M-generic filter G. Let $\vartheta = M \cap On$, then by induction to ϑ we define

$$\mathcal{P}^0(M) = M, \; \mathcal{P}^{\alpha+1}(M) = \mathcal{P}(\mathcal{P}^\alpha(M)) \text{ and } \mathcal{P}^\alpha(M) = \bigcup_{\beta<\alpha} \mathcal{P}^\beta(M), \text{ for } \alpha \text{ limit.}$$

We will have that $M_\alpha[G] \subseteq \mathcal{P}^\alpha(M)$. Also note that $\mathcal{P}^\vartheta(M)$ will be very big but $M = M_\vartheta[G]$ will be a countable subset of $\mathcal{P}^\vartheta(M)$, since there are only countably many M-names.

In subsequent chapters we will see that a generic extension $M = V[G]$ is characterised by the following property

$$(\forall \sigma \in M)\; (\sigma \subseteq V \;\rightarrow\; \exists r \in V \text{ such that } r[G] = \sigma, \text{ hence } \sigma \subseteq \operatorname{rng}(r) \in V),$$

i.e. a generic extension M is determined already by $M_1[G]$.

The next chapter will elaborate examples of forcing notions and the properties of semisets in their generic extensions. This is not a coincidence, since the whole extension is already determined by its semisets, although there of course are new sets, which are not semisets over the ground model.

It will be helpful to know that each semiset has a name of rank 1, i.e.

$$Sm_{V,V[G]} = \{r/_G : r \text{ is of rank } 1\}.$$