1. Set Theory prerequisites

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When we talk about Set Theory we mean the first order predicate theory in a language with equality containing a binary predicate $\in$ with the standard axioms. Formulas describing properties of sets are built up from symbols for variables, predicates ($\in, =$), logical connectives and quantifiers ($\forall, \exists$). If $\phi(x_0, \dots, x_n)$ is such a formula, then $\{\langle x_0, \dots, x_n\rangle : \phi(x_0, \dots, x_n)\}$ is the class defined by this formula. E.g. $V = \{x : x = x\}$ is the class of all sets.

1.1 Axioms of Zermelo-Fraenkel.

1. *Extensionality.* If x and y have the same elements, then $x = y$

$$(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

2. *Pairing.* For any a and b there exists a set $\{a, b\}$ that contains exactly a and b

$$(\forall a)(\forall b)(\exists x)((\forall z)(z \in x \leftrightarrow (z = a \vee z = b)))$$

3. *Separation (Comprehension).* If φ is a formula (with parameter p), then for any set x and p there exists a set $y = \{u \in x : \varphi(u, p)\}$, i.e. a set that contains all those $u \in x$ that satisfy φ. Formally, given a formula φ with u and p as free variables, the formula

$$(\forall p)(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow y \in x \ \& \ \varphi(u, p))$$

is an axiom of set theory.

4. *Union.* For any x there exists a set $y = \bigcup x$, the *union* of all elements of x

$$(\forall x)(\exists y)(z \in y \leftrightarrow (\exists u)(u \in x \ \& \ y \in u))$$

5. *Power Set.* For any x there exists a set $y = \mathcal{P}(x)$, the set of all subsets of x

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\forall u)(u \in z \rightarrow u \in x))$$

6. *Infinity.* There exists an infinite set

$$(\exists x)(\emptyset \in x \ \& \ (\forall z)(z \in x \rightarrow z \cup \{z\} \in x))$$

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7. *Replacement.* The image of a set under a (class) mapping is a set. Formally if $\varphi(x, y)$ is a formula then

$$\big[(\forall x)(\forall y)(\forall z)((\varphi(x,y) \ \&\ \varphi(x,z)) \rightarrow z = y)\big] \rightarrow \big[(\forall a)(\exists b)(y \in b \leftrightarrow (\exists x \in a)\varphi(x,y))\big]$$

is an axiom.

8. *Foundation.* Every nonempty set has an $\in$-minimal element

$$(\forall x)(\exists y)(y \in x \ \&\ (\forall z)(z \in y \rightarrow z \notin x))$$

9. *Choice.* Every set has a choice function

$$(\forall x)(\exists f, \text{function})(\forall z)(z \in x \ \&\ z \neq \emptyset \rightarrow f(z) \in z)$$

COMMENTS.

(i) In the formal statements of the axiom schemas we tacitly assume that the formulas which serve as parameters do not contain “conflicting free variables”.

(ii) Some authors include an axiom postulating the existence of a set, i.e. $(\exists x)(x = x)$. We do not include it here since it is provable from the axioms of predicate logic with equality. Moreover, this is only relevant for fragments which do not include the axiom of infinity.

(iii) Once we have any set, say a, using the third axiom we can prove the existence of the empty set $\{x \in a : x \neq x\}$, customarily denoted as $\emptyset$.

(v) Formally, our language admits variables ranging over sets only, however it is often useful to talk about collections which are too large to be sets, e.g. the universal class V. To overcome this difficulty we shall regard formulas involving class variables as shorthand. Any formula φ naturally determines the collection $F = \{x : \varphi(x)\}$ of all sets satisfying φ. In this situation $x \in F$ is a shorthand for $\varphi(x)$. so, e.g. the formula $(\forall x \in F)\psi$ is shorthand for $(\forall x)(\varphi \rightarrow \psi)$.

(vi) The list of axioms is not the shortest possible, e.g. each instance of the axiom of comprehension follows from an instance of the axiom of replacement. Also pairing can be proved from the other axioms. However since the list is infinite anyway, this makes *no* difference whatsoever.

It is customary to let ZF denote Set Theory with axioms 1.–8. and ZFC Set Theory ZF with choice, i.e. with axiom 9. Occasionally we shall write ZF^- and ZFC^- when we omit axiom 5.

1.2 NOTATION. *Throughout the text we will use the standard notation. For definiteness given a relation* r *and a class* A *we let* $r[A] = \{y : (\exists a \in A)(\langle a, y\rangle \in r\}$. *Similarly for a function* f *we let* $f[A] = \{y : (\exists a \in A)(f(a) = y)\}$, $f^{-1}[A] = \{x : f(x) \in A\}$. *Moreover the domain of a function (or relation)* f *will be denoted by dom*(f) *and the range of* f*, i.e.* $\{y : (\exists x)(f(x) = y)\}$*, will be denoted by rng*(f)*. The corresponding notions for a relation will also be used.*

1.3 DEFINITION. Transitive set. A set a is *transitive* if every element of a is a subset of a, i.e. $(\forall x \in a)(x \subseteq a)$.

1.4 FACT. *Every set* a *has a* transitive closure*, that is, the smallest transitive set containing* a *as an element. It is denoted by tcl*(a)*.*

Proof. (Hint.) By the axiom of union we can define a (class) function F such that $F^0(x) = x$, $F(x) = \bigcup x$ and we let $\mathrm{tcl}(a) = \bigcup_{n<\omega} F^n(\{a\})$, i.e. $\mathrm{tcl}(A) = \{A\} \cup A \cup \bigcup A \cup \bigcup\bigcup A \cdots$ is arrived at by iterating the union operation. See also example 1.32. □

1.5 DEFINITION. Ordinal number. A set α is an ordinal number if

(i) α is a transitive set and

(ii) the $\in$ relation is a linear ordering on α, i.e. for all $\beta, \gamma \in \alpha$ either $\beta \in \gamma$ or $\gamma \in \beta$ or $\beta = \gamma$.

In other words an ordinal number is the set of all smaller ordinal numbers well-ordered by $\in$. The class of all ordinal numbers is denoted by On. We write $\alpha < \beta$ to mean $\alpha \in \beta$.

1.6 DEFINITION (Maximo-lexicographical ordering)**.** The class On^2 of pairs of ordinals can be well ordered in a particularly nice way. We say that $\langle \alpha_1, \beta_1 \rangle \leq_{\mathrm{MLex}} \langle \alpha_2, \beta_2 \rangle$ iff

$$\max\{\alpha_1, \beta_1\} < \max\{\alpha_2, \beta_2\} \quad \vee \quad (\max\{\alpha_1, \beta_1\} = \max\{\alpha_2, \beta_2\} \;\&\; \min\{\alpha_1, \beta_1\} < \min\{\alpha_2, \beta_2\})$$

This ordering is a set-like well order, i.e. for each pair $p \in \mathrm{On}^2$ the class of pairs which are $\leq_{\mathrm{MLex}}$ below p is a set.

1.7 APPROXIMATION OF THE UNIVERSE. We shall show that there are transitive sets which — in a sense — approximate the universe V. The ones most commonly used are the two hierarchies of V_α's and $H(\kappa)$'s.

1.8 DEFINITION. The hierarchy of V_α's. Using transfinite recursion to iterate the power set operation we define V_α for $\alpha \in \mathrm{On}$ as follows:

$$V_\emptyset = \emptyset, \quad V_{\alpha+1} = \mathcal{P}(V_\alpha), \quad V_\alpha = \bigcup_{\beta<\alpha} V_\beta, \text{ for limit } \alpha.$$

1.9 FACT. $V = \bigcup\{V_\alpha : \alpha \in \mathrm{On}\}$.
Note that this fact is equivalent to the axiom of foundation (axiom 8.).

1.10 DEFINITION. Rank. To each set x we assign an ordinal number — its *rank* $\tau(x)$:

$$\tau(x) = \min\{\alpha : x \subseteq V_\alpha\}.$$

It is easily seen that $\tau(x) = \sup\{\tau(y) + 1 : y \in x\}$ for any set x.

1.11 FACT. *Suppose $\alpha < \beta$ are ordinal numbers. Then*

(i) V_α *is a transitive set.*

(ii) $V_\alpha \in V_\beta$.

(iii) $\mathrm{On} \cap V_\alpha = \alpha$*, hence* $\alpha \subseteq V_\alpha$ *but* $\alpha \notin V_\alpha$.

Proof. (i) Given $x \in V_\alpha$, let β be minimal such that $x \in V_\beta$. Clearly $\beta = \gamma + 1 \leq \alpha$. By the definition of V_β we have that $x \in \mathcal{P}(\mathcal{V}_\gamma)$ so $x \subseteq V_\gamma \subseteq V_\beta \subseteq V_\alpha$. (ii) is clear since $x \in \mathcal{P}(x)$ (iii) We show by induction that $\alpha \subseteq V_\alpha$ and $\alpha \notin V_\alpha$: this is clear for $\emptyset$. Suppose we have proven this for all $\beta < \alpha$. If α is limit then by the inductive hypothesis clearly $\alpha \subseteq V_\alpha$ and easily $\alpha \notin V_\alpha$. If $\alpha = \beta + 1$, then $\alpha = \beta \cup \{\beta\}$ and since $\beta \subseteq V_\beta \subseteq V_\alpha$ by the inductive hypothesis and hence $\beta \in V_\alpha$ by the definition of V_α. Similarly $\alpha \notin V_\alpha$. □

1.12 PROPOSITION.

(i) *If $\alpha > \omega$ is a limit ordinal number, then V_α satisfies all axioms of set theory except for the (full) replacement schema. The class of ordinal numbers in the sense of V_α is $\alpha = V_\alpha \cap \mathrm{On}$.*

(ii) *V_ω consists of hereditary finite sets and $(V_\omega, \in)$ is a model of ZF_{fin}, the theory of finite sets. It turns out that it is "equivalent" to the standard model of Peano Arithmetic.*

1.13 DEFINITION. The $H(\kappa)$ hierarchy. Assume AC. If κ is an infinite cardinal number we define $H(\kappa)$ as follows:

$$H(\kappa) = \{x : |\mathrm{tcl}(x)| < \kappa\}.$$

The set $H(\kappa)$ consists of sets which are hereditarily of cardinality less then κ. The universe $H(\omega_1)$ of hereditarily countable sets is sometimes denoted by HC. Note that $H(\kappa)$ is a set for each κ.

In ZFC we again have the following:

1.14 PROPOSITION. (AC)

(i) *Each $H(\kappa)$ is a transitive set.*

(ii) $V = \bigcup\{H(\kappa) : \kappa$ *is a cardinal number*$\}$

1.15 DEFINITION. Recall, that a cardinal number κ is *strongly limit* if if $2^\lambda < \kappa$ for any $\lambda < \kappa$. An uncountable regular strongly limit cardinal κ is called *inaccessible*.

1.16 PROPOSITION. (AC) *If κ is inaccessible or $\kappa = \omega$ then*

$$H(\kappa) = V_\kappa$$

Note that the existence of an inaccessible cardinal is not provable in ZFC, however $H(\kappa) = V_\kappa$ for any fixed point of the function $\beth(\kappa) = 2^{<\kappa}$.

1.17 PROPOSITION. (AC)

(i) *If $\kappa > \omega$ is a regular cardinal, then $(H(\kappa), \in)$ satisfies all axioms of set theory except for the power set axiom 5., i.e. it is a model of ZFC^-. (cf. Proposition 1.12.)*

(ii) *The role of the class of ordinal numbers in the structure $(H(\kappa), \in)$ is played by κ.*

(iii) *If a set b is in $H(\kappa)$, then $\mathcal{P}(b) \subseteq H(\kappa)$.*

(iv) *If $b \in H(\kappa)$ then $\mathcal{P}(b) \in H(\kappa)$ iff $2^{|b|} < \kappa$.*

NOTE. Even thought $H(\kappa)$ need not satisfy the full power-set axiom, (iv) tells us that if we restrict ourselves to sets of "small" cardinality — viz. of cardinality λ such that $2^\lambda < \kappa$ — the power-set axiom holds.

1.18 $H(\kappa)$ AS MODELS OF FRAGMENTS OF ZFC For the purpose of proving consistency results a model of full ZFC is not needed. More precisely: every proof is a finite chain (finite in the meta-theory) of consecutive arguments. Given any finite fragment of ZFC, there are unboundedly many cardinal numbers κ such that $(H(\kappa), \in)$ is a model of this fragment — this is the essence of the reflection principle.

Now of course one does not specify the exact fragment of ZFC one needs. The main point is that it is, at least theoretically, possible to do this. This is the content of the oft seen phrase

"take sufficiently large κ"

which in fact means: take κ large enough so that $(H(\kappa), \in)$ satisfies enough axioms to guarantee all the arguments and constructions which we need.

1.19 WELL-FOUNDED RELATIONS.

1.20 DEFINITION. A binary relation R on a set A is *well-founded* iff each nonempty subset of A contains at least one R-minimal element, that is for any $\emptyset \neq a \subseteq A$ there is an $x \in a$ with

$$\mathrm{Ext}_R(x) \cap a = \emptyset, \quad \text{where} \quad \mathrm{Ext}_R(x) = \{y : \langle y, x\rangle \in R\}.$$

A strict wellordering is an example of a well-founded relation. Well-founded relations are a generalization of wellorders which are themselves a generalization of the $\in$ relation, viz. the axiom of foundation. We shall show (see 1.24), that recursive constructions can be carried out along general well-founded relations much in the same way as along $\in$.

1.21 DEFINITION. Dependent Choice. The axiom of dependent choice is a weaker version of the axiom of choice. It says, that whenever some relation R satisfies

$$(\forall x \in \mathrm{dom}(R))(\exists y)(yRx)$$

then there is an R-descending sequence $\langle x_n : n < \omega\rangle$, i.e. a sequence satisfying $x_{n+1}Rx_n$ for each $n < \omega$. The axiom of dependent choice is abbreviated DC.

1.22 FACT. *If we assume DC then a relation R is well-founded iff there are no infinite R-decreasing chains.*

1.23 WELL-FOUNDED INDUCTION AND RECURSION.

1.24 THEOREM. (Well-founded recursion). *Suppose R is a well-founded and set-like relation on X and G is a mapping defined on V. Then there exists a unique function F defined on X which satisfies*

$$F(x) = G(\{F(y) : yRx\}), \text{ for each } x \in X.$$

1.25 RANK FUNCTION. Every well-founded relation R on X has an associated *rank function* $\rho_R : X \to On$ which is defined using well-founded recursion as:

$$\rho_R(x) = \sup\{\rho_R(y) + 1 : yRx\}, \ \sup\emptyset = 0.$$

If X is a set, then the range of ρ_R is an ordinal number and is called the *height* of R on X. If R is a well-ordering, then the height of R is also called the *order type* of R.

We shall now introduce the often used concept of a tree and show an application of the rank function on ill-founded trees.

1.26 DEFINITION. Tree. A nonempty set T together with a partial order $\leq$ forms a *tree* iff for each $t \in T$ the set of its predecessors $\mathrm{pred}_T(t) = \{s \in T : s < t\}$ is well-ordered by $<$.

(i) Similarly to predecessors, we denote $\mathrm{succ}_T(t)$ the set of all immediate successors of node t in T.

(ii) For each $t \in T$ we define the *height* $h_T(t)$ of t as the order type of $(\mathrm{pred}_T(t), <)$.

(iii) The α-th level of T, denoted by T_α, is defined as $T_\alpha = \{t \in T : h_T(t) = \alpha\}$.

(iv) The height of T is defined to be the rank of $<$ and is equal to $\min\{\alpha : T_\alpha = \emptyset\}$.

(v) A maximal linearly ordered subset of T is called a *branch* of T. The set of all branches of a tree T will be denoted by $[T]$.

By a *subtree* $T' \subset T$ we understand a nonempty subset T' of T which is closed under initial segments, i.e. $(\forall t \in T')$ if $s \leq t$ then $s \in T'$.

1.27 EXAMPLE. Trees of height ω with a single root, i.e. a smallest element, have a nice representation. For a set X define $Seq(X) = {}^{<\omega}X = \bigcup\{{}^{n}X : n < \omega\}$, the set of all finite sequences of elements of X. In case $X = \omega$ we write just Seq. A set $T \subseteq Seq(X)$ closed under initial segments, i.e. if $f \in T$ and $n \in \operatorname{dom} f$ then $f \restriction n \in T$, is called a tree on X. If $X = \kappa$, we say T is a κ-*ary tree*, if $X = 2$, T is *binary tree*. If T is a tree on X then $(T, \subseteq)$ is a tree.

Given a tree T on X we shall identify branches with elements of ${}^{\omega}X$ and let $[T] = \{f \in {}^{\omega}X : (\forall n < \omega)(f \restriction n \in T)\}$ be the set of all branches of T.

1.28 DEFINITION. A tree is called *well-founded* if the reverse order $>$ is well-founded, otherwise it is *ill-founded*. Beware that we take the *reverse* order $>$, since $<$ is well-founded by definition!

1.29 FACT. (ZF)

(i) Assume DC. Then a tree is ill-founded iff it has an infinite branch.

(ii) Assume DC. A tree of height omega with finite levels must have an infinite branch.

(iii) A well-founded binary tree is finite

1.30 EXAMPLE. If T is a well-founded ω-ary tree, then the reverse inclusion relation on T is well-founded, so it has an associated rank function ρ_T. We can then define the rank $r(T)$ of T to be $\rho_T(\emptyset)$.

1.31 EXAMPLE. (DC) If T is a well-founded ω-ary tree, then $r(T) < \omega_1$ and for any $\alpha < \omega_1$ there is a well-founded ω-ary tree T^α with $r(T^\alpha) = \alpha$.

To see that $r(T) < \omega_1$ note that $|T|$ is countable. For the opposite direction we proceed by induction. A tree of rank 1 is just $\{\emptyset\}$. Suppose $\alpha < \omega_1$ and we have constructed a tree T^β of rank β for each $\beta < \alpha$. To construct a tree of rank α, take a tree T of height 2 with $|T_1| = \omega$ and glue the T^β's to the branches of T to get T^α.

The following example will be useful in chapter 6. when we will compare transitive models of ZFC.

1.32 EXAMPLE. Given a set A, let $tcl(A)$ be the set of all $\in$-descending sequences beginning in A. The set $tcl(A)$ together with inclusion is a tree. By the axiom of foundation the height of $(tcl(A), \subseteq)$ is at most ω and the tree has no infinite branches. For convenience we shall add the empty sequence to $tcl(A)$ so that $tcl(A)$ has a single root.

Given any well-founded tree T we can define the following two functions on T:

$$\begin{aligned} f_T(t) &= \{\, f(t') : \ t' \in succ_T(t)\,\}, \\ g_T(t) &= \sup\{\, g(t') + 1 : \ t' \in succ_T(t)\,\}, \text{ where } g(leaf) = 0. \end{aligned}$$

and note that $f_T(s) = \emptyset$ if s is a leaf of T.

1.33 THEOREM. *The tree $tcl(A)$ codes the set A in the sense that $f_{tcl(A)}(\emptyset) = \{A\}$. Moreover given any tree T isomorphic to $tcl(A)$, $f_T(root(T)) = \{A\}$, $g_T(root(T)) = g_{tcl(A)}(\emptyset) = rk(A) + 1 = \min\{\alpha : A \subseteq V_\alpha\}$ and $rng(f_T) = tcl(\{A\})$.*

We now turn to an important theorem that will be useful later on, but first we need a definition.

1.34 DEFINITION. A relation R on a set A is *extensional* if for $a \neq b \in A$ we have that $\mathrm{Ext}_R(a) \neq \mathrm{Ext}_R(b)$, where $\mathrm{Ext}_R(a) = \{x \in A : xRa\}$ (see 1.20).

Note that the axiom of extensionality is equivalent to saying that $\in$ is extensional on V.

1.35 THEOREM. (MOSTOWSKI COLLAPSE). *Suppose R is a well-founded relation on a set A.*

(i) There is a transitive set M and a surjective mapping $F : A \to M$ such that $xRy \to F(x) \in F(y)$.

(ii) Moreover if R is extensional on A, then the mapping F is an isomorphism.

Proof. (hint) Using well-founded recursion, define $F(a) = \{F(b) : b \in A \ \& \ bRa\}$. To show that F is as required, use well-founded induction. □

1.36 ELEMENTARY SUBSTRUCTURES. We will focus on countable substructures of the uncountable structures $(H(\kappa), \in)$ and universal algebras $A = \langle A, \{f_i : i \in I\}\rangle$. We shall tacitly assume that the set of operations is at most countable and each of them is finitary. Suppose we have such an algebra A. A nonempty subset $X \subseteq A$ closed under all these operations determines a subalgebra and whenever $Y \subseteq A$ is infinite, there is an $X \subseteq A$ which is closed under the operations, contains Y and $|Y| = |X|$. We assume the axiom of choice throughout this section.

Let us first recall the classical definition of a club set on ω_1. The cardinal number ω_1 ordered by $\in$ carries a natural topology derived from the order. A club on ω_1 is an unbounded subset of ω_1 which is closed in this topology, i.e. is closed under taking suprema. A standard argument shows that a countable intersection of club sets is again a club set. A subset of ω_1 is called stationary iff it intersects each club set. The following is a classical theorem about stationary subsets, which was subsequently generalized by R. Solovay to higher cardinalities.

1.37 THEOREM (Fodor). *Any stationary subset of ω_1 can be partitioned into ω_1-many disjoint stationary subsets of ω_1.*

In the context of forcing it turns out that the following generalization of club sets to different structures is very convenient.

1.38 DEFINITION. Club. Let A be an infinite set. A family $C \subseteq [A]^\omega$ is *closed unbounded*, or *club* for short, in $[A]^\omega$ if it is

(i) *unbounded,* i.e. $(\forall Y \in [A]^\omega)(\exists X \in C)(Y \subseteq X)$ and

(ii) *closed,* i.e. for any increasing chain $\{X_n : n < \omega\}$ of elements of C the union $\bigcup\{X_n : n < \omega\}$ is again an element of C.

NOTE. In our definition we have only assumed that A is infinite. If it is countable, however, the notion is not very interesting, in particular the singleton $\{A\}$ is a club.

NOTE. Observe that if C is a (classical) club on ω_1 then it is also a club in $[\omega_1]^\omega$ (more precisely $\{\alpha : \alpha \in C \ \& \ \omega \leq \alpha\}$ is a club in $[\omega_1]^\omega$).

1.39 FACT. *Suppose A is an infinite set. It is relatively easy to prove that the intersection of a countable system of clubs in $[A]^\omega$ is again a club in $[A]^\omega$. It follows that the system of clubs in $[A]^\omega$ generates a σ-complete filter.*

The following proposition is an algebraic version of the well-known Löwenheim - Skolem theorem from model theory.

1.40 PROPOSITION. *Suppose* $\mathcal{A} = \langle A, \{f_n : n < \omega\}\rangle$ *is an algebra with infinite underlying set* A. *Then the family of all countable subalgebras forms a club in* $[A]^\omega$.

The converse of the proposition also holds (see e.g. [Jec02]):

1.41 PROPOSITION. *Suppose that* C *is a club in* $[A]^\omega$, *then there are operations* $\{f_n : n < \omega\}$ *on* A *such that* C *contains all countable subalgebras of* A *with given operations.*

Having defined closed unbounded families, we can define stationary families.

1.42 DEFINITION. Stationary set. A family $S \subseteq [A]^\omega$ is *stationary* if it intersects each club, i.e. if C is a club in $[A]^\omega$, then $S \cap C \neq \emptyset$.

It follows from propositions 1.40, 1.41 that:
Consider now the structure $(H(\kappa), \in)$.

1.43 DEFINITION. Elementary substructure. Let $X \subseteq H(\kappa)$ be a countable set. We shall say that X is an *elementary substructure* of $H(\kappa)$, denoted $(X, \in) \preceq (H(\kappa), \in)$, if for each property expressed by a formula of Set Theory $\varphi(\nu_1, \ldots, \nu_n)$ with free variables $\nu_1, \ldots, \nu_n$, the following holds:

$$(\forall x_1, \ldots, x_n \in X)((X, \in) \models \varphi[x_1, \ldots, x_n] \equiv (H(\kappa), \in) \models \varphi[x_1, \ldots, x_n])$$

This implies, in particular, that sentences (i.e. closed formulas) of Set Theory are valid in $(X, \in)$ iff they are valid in $(H(\kappa), \in)$.

RELATIVITY PRINCIPLE. The concept of an elementary substructure is akin to the Galileo's relativity principle in physics. This principle states that, using physical experiments, we cannot decide which frame of reference we are in. Similarly, using Set Theoretical properties, we cannot distinguish whether we are in the elementary substructure or the superstructure. In other words, the validity of statements about elements of X cannot distinguish between X and $H(\kappa)$.

1.44 FACT. *For any* $X \preceq H(\kappa)$ *the following basic facts are true:*

(i) $\omega \subseteq X$ *since each natural number is definable in* $H(\kappa)$ *and*

(ii) $\omega \in X$, *since* ω *is also definable in* $H(\kappa)$,

more generally

(iii) If $Y \in X$ *and* Y *is countable, then* $Y \subseteq X$.

NOTE. Countable elementary substructures typically are not transitive sets, and for $\kappa > \omega_1$ are provably not transitive. It is hence perfectly possible that some uncountable set is an *element* of the countable substructure. In fact, $\omega_1 \in X$ whenever X is an elementary substructure of $H(\kappa)$ for $\kappa > \omega_1$.

As with algebras, we have the following often used and important theorem.

1.45 THEOREM. *Suppose* $\kappa > \omega$. *Then*

(i) for any countable subset $Y \subseteq H(\kappa)$, *there is an* $X \preceq H(\kappa)$ *such that* $Y \subseteq X$ *and*

(ii) the set of countable elementary substructures of $H(\kappa)$ *forms a club in* $[H(\kappa)]^\omega$.

Stationary sets on $[A]^\omega$ are important in the theory of proper forcing. For more information on elementary substructures and their usage e.g. in topology see [Dow95]. Here we mention an application in set theory

1.46 THEOREM. *It κ is an inaccessible cardinal, then $V_\kappa = H(\kappa)$ and V_κ is a model of ZFC. Moreover there exists a cardinal number $\lambda < \kappa$ with countable cofinality such that V_λ is a model of ZFC.*

Proof. By induction find $\alpha_n < \omega$ and elementary substructures $X_n \preceq H(\kappa)$ such that $X_n \subseteq V_{\alpha_n} \subseteq X_{n+1}$. Then $\lambda = \sup\{\alpha_n : n < \omega$ is as required. □

1.47 ELEMENTARY SUBSTRUCTURES AS PROOF TOOLS. If we want to prove that a given set a has property φ the following argument is sometimes useful. We consider a "sufficiently large κ" such that $a \in H(\kappa)$. Then, taking an elementary substructure $(X, \in, a) \prec (H(\kappa), \in, a)$, it is sometimes easier to prove that

$$(X, \in, a) \vDash \varphi(a),$$

By elementarity we get $(H(\kappa), \in, a) \vDash \varphi(a)$ and, since κ was "sufficiently large" then also ZFC $\vdash \varphi(a)$.

1.48 INTERPRETATIONS & FINITARY CONSISTENCY PROOFS. We shall now sketch an effective method for proving relative consistency. This method was used by Gödel to prove the relative consistency with ZF of CH and the Axiom of choice.

First we introduce the notion of *interpretation* of the language of Set Theory in Set Theory itself. The language of Set Theory is determined by two basic binary predicates: equality ($=$) and membership ($\in$). Using these two predicates we may define, as is customary, further predicates (e.g. $\cup, \emptyset, \omega, \mathbb{R}, \ldots$).

To interpret this language in Set Theory, we need to define a class U and two binary relations, which we shall suggestively call $=^*$ and $\in^*$. We also require that the following basic conditions are satisfied, i.e. provable in Set Theory,

(a) $(\exists x)U$,

(b) $=^*$ is an equivalence relation on $\{x : U(x)\}$ and

(c) $(\forall x, y, z)(U(x) \,\&\, U(y) \,\&\, U(z) \to ((x \in^* y \,\&\, x =^* z \to z \in^* y) \,\&\, (x \in^* y \,\&\, y =^* z \to x \in^* z)))$.

Then we can translate any formula φ of Set Theory into the corresponding formula φ^* — the interpretation of φ — by replacing all occurrences of $=$ with $=^*$, $\in$ with $\in^*$ and $\forall x$ with $(\forall x \in U)$. The precise definition is, of course, by induction on the complexity of φ:

If φ is $x \in y$, then φ^* is $x \in^* y$,

if φ is $x = y$ then φ^* is $x =^* y$ and

if φ is $\neg(\psi)$ then φ^* is $\neg(\psi^*)$,

if φ is $\psi \to \chi$ then φ^* is $\psi^* \to \chi^*$ and

if φ is $(\forall x)\psi$ then φ^* is $(\forall x)(U \to \psi^*)$.

We shall say that $(U, =^*, \in^*)$ is a *syntactical model* of a theory T if any axiom of T translates into a provable formula. We say that the model *satisfies* a sentence φ if φ^* is provable.

1.49 METATHEOREM. *Suppose $(U, =^*, \in^*)$ is a syntactical model of Set Theory which satisfies some sentence φ. Then φ is consistent with Set Theory.*

Proof. We assume Set Theory is consistent, otherwise the theorem is clear. Given a sentence ψ provable in Set Theory by induction on the length of its proof we show that it is satisfied by $U, =^*, \in^*$. Now if φ were inconsistent, then $\neg\varphi$ would be provable so also $(\neg\varphi)^* \equiv \neg(\varphi^*)$ would be provable. Since we assumed φ^* is provable we have shown that Set Theory itself is inconsistent a contradiction. □

A common special case of interpretations is that we only define the class U and let $=^*, \in^*$ be just $=, \in$. In that case the translated formula φ^* is often denoted as φ^U.

1.50 EXAMPLE. If $U \neq \emptyset$ is a transitive class and φ is the axiom of extensionality, then φ^U is provable. Let us call this interpretation transitive. A formula with φ with free variables $x_0, \dots, x_n$ is called *absolute* for an interpretation if

$$(\forall x_0, \dots, x_n \in U)(\varphi^* \leftrightarrow \varphi).$$

Transitive interpretations have the advantage that many formulas are absolute. For example $z = \{x, y\}$ is absolute so for any $z \in U$ we have "(is a pair)^U" iff z is a pair. This allows us to simplify many arguments in the case of transitive interpretations.

1.51 EXAMPLE (Gödel)**.** Let L be the class of constructible sets (see e.g. [Jec02]). L is a transitive class.Then if φ is an axiom of ZF, then φ^L is provable. Moreover GCH^L and AC^L are provable. It follows that the continuum hypothesis and the axiom of choice are both consistent with ZF.

1.52 SEMISETS. We now introduce a working term which will be used later on. Please keep in mind that this concept is not generally known in the wide Set-Theory circles. Nevertheless it is a simple notion and appears to be quite handy for generic extension. For more details and in-depth discussion of the concept see [VH72]. For our purposes we shall not assume — or need — any knowledge of this book.

As a motivation for the definition consider the axiom of power set. Conventionally it is understood to say that the collection of all parts of a set forms a set. The following interpretation, which leads to the definition of a semiset, is equally valid: The axiom of powerset says that the collection of all parts of a set *which are themselves a set* forms a set.

1.53 DEFINITION. Semiset. A *semiset* is a part of a set. A semiset which is not a set is *proper*.

Note that each set is a semiset and in our Set Theory there are no proper semisets. We will now describe two natural scenarios which lead to this notion.

MOTIVATING SEMISETS.

I. Take an infinite cardinal λ and consider $H(\kappa)$ where $\kappa = (2^\lambda)^+$. We know that $(H(\kappa), \in)$ is a model of ZFC^- (see Proposition 1.17, (i)). In our special case, $\mathcal{P}(\lambda) \in H(\kappa)$ so the power set axiom holds for λ. Let X be a countable elementary substructure of $H(\kappa)$ with $\lambda \in X$.

Then $(X, \in)$ is a model for ZFC^-, $\lambda, \mathcal{P}(\lambda) \in X$ and $\in$ is a well-founded extensional relation on X. Applying the Mostowski collapse (see Theorem 1.35) to obtain a transitive set M and an isomorphism F of $(X, \in)$ and $(M, \in)$. $(M, \in)$ is again a countable model of ZFC^-. All subsets of $F(\lambda)$ which are not elements of M are proper semisets from the point of view of M.

II. Another context, where semisets arise, are nonstandard models of ZFC. For example consider the ultrapower $V^\omega/\mathcal{U}$ where $\mathcal{U}$ is a nontrivial ultrafilter on ω. In this structure there are proper semisets which are even parts of a natural number: the class of standard natural numbers is an example of such a semiset. It has to be proper, since it has no maximal element. Conversely there are parts of the natural numbers with no minimal elements, so, again, they must be proper semisets.