The Random Field Ising Model (revisited)

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in collaboration with
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This talk

Can RFIM be glassy?

RFIM on random graph with fixed magnetization

Random graph bi-partitioning.
Definition of RFIM

\[ \mathcal{H} = - \sum_{ij} J_{ij} S_i S_j - \sum_i H_i S_i \]

\[ \forall i, j \quad J_{ij} \geq 0 \quad \quad S_i \in \{-1, +1\} \]

★ General topology (lattice, graph)
★ General distribution of random field

bimodal: \[ [\delta(H_i - H_R) + \delta(H_i + H_R)]/2 \]
Gaussian: \[ N(0, H_R) \]
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bimodal: \[ \frac{\delta(H_i - H_R) + \delta(H_i + H_R)}{2} \]

Gaussian: \( N(0, H_R) \)
Some known results

- Today’s talk of M. Aizenman: lower critical dimension 2, disorder relevant; failure of dimensional reduction.

- Exact solution on fully connected graph (Schneider, Pytte’77)

Ferromagnetic transition:
- 2nd order for gaussian field
- 1st order for bimodal field
Random Field Ising Model: Dimensional Reduction or Spin-Glass Phase?

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Abstract
The stability of the random field Ising model (RFIM) against spin glass (SG) fluctuations, as investigated by Mézard and Young, is naturally expressed via Legendre transforms, stability being then associated with the non-negativity of eigenvalues of the inverse of a generalized SG susceptibility matrix. It is found that the signal for the occurrence of the SG transition will manifest itself in free-energy fluctuations only, and in the free energy itself. Eigenvalues of the inverse SG susceptibility matrix are then investigated by the Rayleigh Ritz method which provides an upper bound. Coming from the paramagnetic phase on the Curie line, one is able to use a virial-like relationship generated by the single unit length ( \(D<6\); in higher dimension a new length sets in, the inverse momentum cut off). Instability towards a SG phase being probed on pairs of distinct replicas, it follows that, despite the repulsive coupling of the RFIM the effective pair coupling is attractive (at least for small values of the parameter \(g\tilde{A}\), \(g\) the coupling and \(\tilde{A}\) the effective random field fluctuation). As a result, "bound states" associated with replica pairs (negative eigenvalues) provide the instability signature. Away from the Curie line, the attraction is damped out till the SG transition line is reached and paramagnetism restored. In \(D<6\), the SG transition always precedes the ferromagnetic one, thus the domain in dimension where standard dimensional reduction would apply (on the Curie line) shrinks to zero.

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Glassy transition in the three-dimensional random-field Ising model

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(Received 3 June 1994)

The high-temperature phase of the three-dimensional random-field Ising model is studied using the replica symmetry-breaking framework. It is found that, above the ferromagnetic transition temperature $T_f$, a glassy phase appears at intermediate temperatures $T_f < T < T_b$ while the usual paramagnetic phase exists for $T > T_b$ only. The correlation length at $T_b$ is computed and found to be compatible with previous numerical results.
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**Interactions of several replicas in the random field Ising model**

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(Received 28 July 2000)

**Abstract**

The replicated field theory of the random field Ising model involves the couplings of replicas of different indices. The resulting correlation functions involve a superposition of different types of long distance behaviours. However, the \(n=0\) limit allows one to discuss the renormalization group properties in spite of this phenomenon. The attraction of pairs of replicas is enhanced under renormalization flow and no stable fixed point is found. Consequently, an instability occurs in the paramagnetic region, before one reaches the Curie line, signalling the onset of replica symmetry breaking.

**PACS**

75.10.Nr - Spin-glass and other random models.
05.50.+q - Lattice theory and statistics (Ising, Potts, etc.).

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Replica symmetry breaking in random-field systems

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Received 5 January 1987

The conventional mean-field theory of random-field systems is, for large but finite dimensionalities d, found to be in error. For larger applied fields, a separate glassy phase appears in the phase diagram of the bond-diluted antiferromagnet. For large d, the diluted antiferromagnet can be mapped on the antiferromagnet with Gaussian randomness and the phase diagram of this model is shown to have a de Almeida–Thouless line, marking the onset of the glassy phase by replica symmetry breaking. In the limit d→∞ we recover conventional mean-field theory.

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Mean-field glassy phase of the random-field Ising model

Abstract

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Received 20 July 2001; revised 15 April 2002; published 1 July 2002

The emergence of glassy behavior of the random-field Ising model (RFIM) is investigated using an extended mean-field theory approach. Using this formulation, systematic corrections to the standard Bragg-Williams theory can be incorporated, leading to the appearance of a glassy phase, in agreement with the results of the self-consistent screening theory of Mezard and Young. Our approach makes it also possible to obtain information about the low-temperature behavior of this glassy phase. We present results showing that within our mean-field framework, the hysteresis and avalanche behavior of the RFIM is characterized by power-law distributions, in close analogy with recent results obtained for mean-field spin-glass models.

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Spin glass phase

- (Static) Replica symmetry breaking
- Non-trivial overlap function
- De Almeida-Thouless condition (AT line)
- Droplet picture
- Experiments: Cusp in non-linear susceptibility
Spin glass phase

(Static) Replica symmetry breaking
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De Almeida-Thouless condition (AT line)
Droplet picture
Experiments: Cusp in non-linear susceptibility

Divergence of the spin glass susceptibility:

\[ \chi_{SG} = \frac{1}{N} \sum_{ij} \langle S_i S_j \rangle_c^2 \]
For any lattice, any random fields:

\[ \forall ij \quad \langle S_i S_j \rangle_c \geq 0 \]
Proof by induction

\( N = 2 \)

\[ \langle S_1 S_2 \rangle_c = \frac{8}{Z^2} \sinh (2\beta J_{12}) \geq 0 \]

\( N \Rightarrow N + 1 \)

\[ h_i^{(N+1)} = h_i^{(N)} + S_{N+1} J_{N+1,i} \]

\[ \langle S_i \rangle^{(N+1)} \equiv 2\alpha - 1 \]

\[ \langle S_{N+1} S_i \rangle^{(N+1)}_c = 2\alpha (1 - \alpha) (\langle S_i \rangle^{(N)}_+ - \langle S_i \rangle^{(N)}_-) \geq 0 \]

\[ \langle S_i S_j \rangle^{(N+1)}_c = \alpha \langle S_i S_j \rangle^{(N)}_{c,+} + (1 - \alpha) \langle S_i S_j \rangle^{(N)}_{c,-} + \]

\[ + \alpha (1 - \alpha) (\langle S_i \rangle^{(N)}_+ - \langle S_i \rangle^{(N)}_-) (\langle S_j \rangle^{(N)}_+ - \langle S_j \rangle^{(N)}_-) \geq 0 \]

q.e.d.
\[ \chi_{\text{ferro}} = \frac{1}{N} \sum_{ij} \langle S_i S_j \rangle_c \geq \frac{1}{N} \sum_{ij} \langle S_i S_j \rangle_c^2 = \chi_{SG} \]
\[ \chi_{\text{ferro}} = \frac{1}{N} \sum_{ij} \langle S_i S_j \rangle_c \geq \frac{1}{N} \sum_{ij} \langle S_i S_j \rangle_c^2 = \chi_{\text{SG}} \]

\[ \chi_{\text{ferro}} \geq \chi_{\text{SG}} \]

- Any topology (lattice, graph)
- Any distribution of random field
- Any non-negative interactions
No glass out of the critical ferro temperature

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No glass out of the critical ferro temperature
This talk

- Can RFIM be glassy?
- RFIM on random graph with fixed magnetization
- Random graph bi-partitioning.
RFIM at fixed magnetization

- Lattice gas models in grand-canonical ensemble.
- Electron (Coulomb) glass - half filling = zero magnetization.
- Graph partitioning - very important optimization problem
- Does fixing the magnetization allow a glassy phase?
RFIM at fixed magnetization

- Lattice gas models in grand-canonical ensemble.
- Electron (Coulomb) glass - half filling = zero magnetization.
- Graph partitioning - very important optimization problem
- Does fixing the magnetization allow a glassy phase?

Let's study this on sparse random graphs
Bethe–Peierls = Belief Propagation = Replica Symmetric = Liquid Solution

\[ h^{i \rightarrow j} = H_i + \frac{1}{\beta} \sum_{k \in \partial i \setminus j} \tanh^{-1} \left[ \tanh (\beta J_{ki}) \tanh (\beta h^{k \rightarrow i}) \right] \]

\[ m = \frac{1}{N} \sum_i \tanh \left[ \beta H_i + \sum_{j \in \partial i} \tanh^{-1} (\tanh \beta J_{ji} \tanh \beta h^{j \rightarrow i}) \right] \]
Random graph (Bethe lattice) phase diagram

Gaussian field $N(0, H)$

3D values

$T_c \approx 4.5$

$H_c \approx 2.3$
How to fix the magnetization?

Standard: Via external (uniform) magnetic field.

\[ \mathcal{H}_h = \mathcal{H} - H_m \sum_i S_i \]

\[ f(m) = f_h(H_m) + H_m m \]

\[ \frac{\partial f_h(H_m)}{\partial H_m} = -m \]
How to fix the magnetization?

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\[ f(m) = f_h(H_m) + H_m m \]

\[ \frac{\partial f_h(H_m)}{\partial H_m} = -m \]

Only the convex envelope of \( f(m) \)!

Concave part of \( f(m) \) corresponds to unstable fixed points.
“All” is possible with BP

BP can be iteratively forced to converge into the unstable fixed point.

(Di, Montanari, Urbanke’04, Mora, Mezard’06)

\[
h^{i \rightarrow j} = H_m + H_i + \frac{1}{\beta} \sum_{k \in \partial i \setminus j} \tanh^{-1} \left[ \tanh (\beta J_{ki}) \tanh (\beta h^{k \rightarrow i}) \right]
\]

\[
m = \frac{1}{N} \sum_i \tanh \left[ \beta (H_m + H_i) + \sum_{j \in \partial i} \tanh^{-1} (\tanh \beta J_{ji} \tanh \beta h^{j \rightarrow i}) \right]
\]
Free energy at fixed m

\[ H_R = N(0,1) \]
Stability towards RSB

- RS unstable = BP ceases to converge
- Compute "Lyapunov exponents" $\lambda$ for convergence of BP.
- Generalization of the de Almeida-Thouless approach to random graphs

$\lambda > 1$ \quad RS unstable

$\Rightarrow \chi_{SG} \rightarrow \infty$
Spin Glass Stability

\[ H_R = N(0,1) \]
General picture

Glassy phase may exist in the concave part of \( f(m) \)
RFIM
Bethe lattice
Gaussian fields
RFIM
Bethe lattice
Gaussian fields
RFIM
Bethe lattice
bimodal fields
Finite dimension

Maxwell construction = phase coexistence

But signatures of the glassy states may survive ...
Finite dimension

Maxwell construction = phase coexistence

But signatures of the glassy states may survive ...
This talk

- Can RFIM be glassy?
- RFIM on random graph with fixed magnetization
- Random graph bi-partitioning.
Graph bi-partitioning

Split nodes of the graph into two groups of a given size in such a way that the number of edges between the two groups is the smallest possible.

Ground state of the Ising Model at fixed magnetization
Application of statistical mechanics to NP-complete problems in combinatorial optimisation

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Received 4 September 1985

Abstract. Recently developed techniques of the statistical mechanics of random systems are applied to the graph partitioning problem. The averaged cost function is calculated and agrees well with numerical results. The problem bears close resemblance to that of spin glasses. We find a spin glass transition in the system, and the low temperature phase space has an ultrametric structure. This sheds light on the nature of hard computation problems.
Phase Diagram of Erdos-Renyi bi-partitioning

\[ c_s(m = 0) = 2 \ln 2 \quad \text{and} \quad c_d(m = 0) = 1.47 \]
Bi-partition size
E.g.: Random 3-regular graphs

Remark: Random regular graphs, $m=0$, RSB equations at zero temperature equivalent to those of a spin glass.
in the Hamiltonian. The model can now be solved using standard techniques. Introducing Lagrange multipliers to decouple the quadratic terms we find that the constraint is irrelevant at $T = 0$, and the equivalence with the $\mathbb{Z}_2$ spin glass becomes exact in this limit, as shown in appendix 1. Alternatively we can argue that, since the ground states of the $\mathbb{Z}_2$ spin glass do not have finite magnetisation per spin, lifting the constraint at this stage will not affect $C/N$ for large $N$. The largest contribution to the free energy due to ferromagnetic fluctuations has already been eliminated as in (3.5). Using the known value of the zero temperature energy of the $\mathbb{Z}_2$ spin glass $U_0$ [2] we have

$$C(p) = \frac{1}{4} N^2 p + \frac{1}{2} U_0 N^{3/2} \left[ p(1-p) \right]^{1/2} = \frac{1}{4} N^2 p - 0.38 N^{3/2} \left[ p(1-p) \right]^{1/2}. \quad (3.9)$$
Graph bipartitioning and the Bethe spin glass


Abstract. The problem of the equipartitioning of a random graph of fixed finite valence is studied by comparison with a ferromagnetic Bethe lattice with random boundary conditions. The simplicity of recursion relations for effective fields due to descendants on Bethe lattices provides simple approximations for the optimal cost, in quite good agreement with simulations.

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LETTER TO THE EDITOR

Graph partitioning and dilute spin glasses: the minimum cost solution

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Abstract. We calculate the ground-state cost function/energy for graph bipartitioning and spin glasses on networks with fixed finite valences, allowing a continuous distribution for the effective field. Compared with previous calculations using a field distribution in integral multiples of the coupling strength J, the ground-state energy is higher by less than one percent, although the distribution itself is altered drastically.
Conjecture

For random regular graphs in the leading order:

\[
\text{min bisection} = \# \text{ of edges} - \text{max-cut size} \]

\[
\# \text{ of violated edges in the ground state of } \pm J \text{ spin glass}
\]

Does not hold for nonzero magnetization nor non-regular graphs.